BOOK REVIEWS

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Riemannian geometry, by Wilhelm Klingenberg, de Gruyter Studies in Mathematics, Vol. 1, Walter de Gruyter & Co., Berlin, 1982, x + 396 pp., $49.00. ISBN 3-1100-8373-5

The basic concepts of Riemannian geometry have become useful in a surprising variety of mathematical subjects. The terminology of manifolds, bundles, Riemannian metrics, and connections has become a *lingua franca* over much of partial differential equations, mathematical physics, and algebraic geometry, among other fields. This increasingly widespread use of the terminology and methods of geometry has tended to obscure the fact that Riemannian geometry as such is a subject with a quite precisely focused program, namely, to determine how the topology of a manifold is influenced by the local properties of its metric structure. These local properties are usually formulated in terms of curvature; this formulation is justified by the theorem of E. Cartan that the curvature tensor and its covariant derivatives of all orders at a point determine the formal Taylor expansion of the metric at that point. The program of obtaining global topological information from local metric information actually applies only to complete Riemannian manifolds, i.e., those that are complete as metric spaces when the distance between points is defined to be the infimum of the Riemannian-metric arc length of curves joining the two points. This restriction to complete manifolds, long a standard one in the subject, has been given explicit justification by a result of M. Gromov that a noncompact manifold admits noncomplete Riemannian metrics with essentially arbitrary curvature behavior.

A complete Riemannian manifold has the property that the geodesics (curves that locally minimize distance) emanating from a fixed but arbitrary point cover, taken together, the whole manifold. It follows that understanding the behavior of these geodesics completely would yield total information about the global structure of the manifold itself. This rather vague description can be given real mathematical substance; and from this general idea, a body of results has been obtained that is the central part of Riemannian geometry. These results are the subject of this book.

To describe the contents of the book in more detail, it is necessary to recall the concept of Riemannian sectional curvature. This concept is a natural extension of the older idea of the Gauss curvature of a surface. If M is a Riemannian manifold, p a point of M, and P a two-dimensional subspace of the tangent space to M at p, then the arc-length parameter geodesics through p with tangent vector lying in P form near p a C^∞ two-dimensional submanifold of M. The Gauss curvature of this surface (with its metric induced from that of M) at p is by definition the sectional curvature of the 2-plane P. It is an algebraic observation that the sectional curvature function at a point p determines uniquely the curvature tensor at p, or, equivalently, it determines the metric near p up to and including the second order part of its Taylor expansion at p. It turns out that the most geometrically natural way of specifying local metric restrictions on a Riemannian manifold, so as to yield global topological information, is to make some hypothesis about sectional curvature.

The most obvious question of this nature is: what are the complete manifolds of constant sectional curvature? Because multiplication of the metric by a positive constant λ divides the sectional curvature by the same constant, this question reduces to what are the manifolds of constant sectional curvature $+1$, -1, or 0—the answers being, if the manifold is assumed simply connected, the unit spheres (of each dimension), the hyperbolic spaces, and the euclidean spaces. [The non-simply-connected constant curvature manifolds are quotients of these; the determination of all possible quotients is highly nontrivial, however, and in fact has been carried out completely only for the $+1$ case, by J. Wolf.]

Once the constant curvature situation is understood, it is natural to investigate the possibilities of almost constant curvature. With the behavior under scaling (as noted) in view, the natural form of this restriction becomes, in the positive curvature case, $0 < \delta \leqslant$ sectional curvature $\leqslant 1$ for some δ. The corresponding condition for the nonpositive curvature cases is largely subsumed by the classical theorem of v. Mangoldt-Hadamard-Cartan that a complete simply connected Riemannian manifold of nonpositive curvature is diffeomorphic to euclidean space. (Of course, the question of what types of quotients occur to yield the non-simply-connected manifolds of nonpositive curvature arises; this will be further discussed momentarily.) The first result in the positive curvature case was the theorem of S. Myers, extending (much) earlier work of Bonnet: A complete Riemannian manifold with $\delta <$ sectional curvature for some $\delta > 0$ is necessarily compact. (Myers' result actually requires only a positive lower bound on the Ricci curvature; rather strangely, Ricci curvature is not mentioned in this book.) The paraboloid of revolution in $\mathbf{R}^3$ shows that a positive lower bound, not just positivity itself, is needed to imply compactness.

The Bonnet-Myers theorem and the v. Mangoldt-Hadamard-Cartan theorem illustrate the fundamental intuitions concerning how curvature controls geodesic behavior: (more) positivity of curvature makes geodesics diverge less rapidly, and more negativity of curvature makes them diverge more rapidly. In particular, the Bonnet-Myers proof compares the geodesics on the manifold with sectional curvature $> \delta$ to those on the sphere of curvature equal to δ everywhere; by the intuitive principle noted, one would expect the manifold's geodesics to converge sooner than the sphere's. The precise (and true) version is that the manifold must have diameter $\leqslant$ the diameter of the sphere. In the case of the v. Mangoldt-Hadamard-Cartan theorem, one compares the manifold to euclidean space, and the conclusion is that geodesics from a fixed point of the manifold diverge forever, as in euclidean space, forcing the manifold to be topologically trivial. Caution is needed here: strictly speaking, the comparison principle applies only to varying a geodesic through nearby ones. In the nonpositive curvature case, simple connectivity is utilized to extend the comparison to give global results. The intuitive comparison principle just described is of course given precise form via the ideas of second variation and Jacobi fields and the Rauch Comparison Theorem.

A compact simply-connected manifold of positive sectional curvature need not be topologically a sphere. The complex projective spaces of complex dimension at least two admit Riemannian metrics with sectional curvature varying between $1/4$ and 1. Numerous other examples of positive curvature compact manifolds are known (quaternionic projective spaces and certain other homogeneous spaces, and some nonhomogeneous examples recently discovered by J. Eschenburg). One of the outstanding results of modern geometry is that no nonspherical example can satisfy $1/4 <$ sectional curvature $\leqslant 1$. Precisely, if M is a complete simply-connected Riemannian manifold with $1/4 <$ sectional curvature $\leqslant 1$, then M is homeomorphic to a sphere. The proof of this result, originally obtained by M. Berger for even-dimensional manifolds, and by W. Klingenberg (the present author) for odd-dimensional ones, involves understanding of geodesic behavior going far beyond simple application of comparison principles. Subtle and ingenious argument is required, although the comparison principles are still the basis of the arguments.

Noncompact manifolds of nonnegative curvature can also be well understood by analysis of geodesic behavior. The results are that a complete noncompact manifold of everywhere positive sectional curvature is diffeomorphic to a euclidean space (proved by D. Gromoll and W. Meyer); and a noncompact complete manifold of nonnegative curvature is diffeomorphic to the total space of a vector bundle over a compact manifold of nonnegative curvature (proved by J. Cheeger and D. Gromoll, after the Gromoll-Meyer result). These results are again based on the comparison principle, in the following form: In a manifold of nonnegative curvature, the sphere of radius r around an arbitrary point p $(= \{q \mid \text{distance}(p, q) = r\})$ is less convex than the euclidean sphere of radius r. In general, such spheres need not be smooth, so the precise interpretation of this principle involves some further technical considerations. In application, this principle implies that if $C\colon [0, \infty) \to M$ is a geodesic ray (i.e., C is minimizing between any two of its points), then

$$M - \bigcup_{t>0} \{q \mid \text{distance}(q, C(t+r)) \leqslant t\}$$

is convex for each $r \geqslant 0$. The construction of such convex sets enables one to show, by Morse Theory, that the topology of the manifold is entirely determined by that of certain compact convex subsets, from which determination the theorem is then deduced.

The global theorems just described (the v. Mangoldt-Cartan-Hadamard Theorem, the Sphere Theorem of Berger and Klingenberg, and the theorems of Gromoll-Meyer and Cheeger-Gromoll) are the main goals of the first two parts (chapters) of this three-part book. The first chapter presents basic material on manifolds, covariant derivatives, local behavior of geodesics, and Jacobi fields. The second chapter presents more detailed analysis of geodesics—cut locus, completeness, Morse theory, Rauch and Toponogov comparison theorems—and the proofs of the theorems. The Gromoll-Meyer and Cheeger-Gromoll theorems are in fact proved only up to homotopy, with the rather messy details of the diffeomorphism statements summarized briefly with references. These first two parts of the book together give a clear and still quite concise (255 pages) presentation of the essentials of global Riemannian geometry.

The third chapter of the book pursues a related, but somewhat different direction, namely, the consideration of the geodesics of a Riemannian manifold as a dynamical system, the geodesic flow. In more detail: on the tangent bundle of a complete Riemannian manifold, there is a one-parameter family of diffeomorphisms ϕ_t defined by, for V a tangent vector at p in M, $\phi_t(V) =$ the tangent vector at time t of the geodesic with intial ($t = 0$) point p and initial tangent vector V. This flow is of great interest as a fundamental example for the general theory of dynamical systems; also, information about the geodesic flow can be used to prove results about the Riemannian geometry of manifolds, often in unexpected ways. The study of the geodesic flow is particularly relevant to the geometrically natural question of the existence and properties of smoothly closed geodesics, such geodesics being exactly the closed orbits of the flow. This question is under active investigation in contemporary research, as it has been continuously at least since the work of Poincaré on the subject. Although this part of the book stops, in general, short (presumably in the interests of brevity) of contemporary results, the material presented does provide an excellent introduction to the basic modes of thought about the subject. What makes the presentation especially useful and attractive is that many results of a concrete, detailed nature are presented—for instance, a detailed analysis of geodesics on ellipsoids in $\mathbf{R}^3$, and a concrete proof without use of Hilbert manifolds of the Lyusternik-Schnirelmann theorem that there are three simple closed geodesics on a surface of genus zero. There is also an extended discussion of compact manifolds of negative curvature including Preismann's theorem that every abelian subgroup of the fundamental group of such a manifold is cyclic and recent generalizations of this result, and an introduction to current results on the geodesic flow of such manifolds (topological transitivity is proved). This abundance of concrete results and examples enables the reader to obtain some intuitive insights that are a very useful preparation for, and adjunct to, the powerful, but more formal, less intuitive methods (Hilbert manifolds, ergodic theory, general dynamical systems theory) of contemporary research.

This book is outstanding in its choice of material. It offers the reader an opportunity to progress from minimal prerequisites to substantial competence in Riemannian geometry. The formal prerequisites are truly minimal—almost nothing is required except a knowledge of calculus and, occasionally, Hilbert spaces and homotopy theory, to the extent of the definition of the fundamental group and higher homotopy groups. Realistically, though, a reader will need some considerable prior familiarity with manifolds, for without it the introductory material (the first half of Chapter 1) will seem very terse and overly formal. It is in fact quite formal in any case, because the author wants to allow manifolds modeled on separable Hilbert spaces, and this allowance necessitates more formal care than is required in the finite-dimensional situation. Once past this slightly forbidding initial development, however, the reader will find a mixture of the general and the concrete that to my mind is essentially an ideal presentation of the subject. For me, and I imagine for all geometers, there is no doubt that the material in this book is some of the "right stuff", in T. Wolfe's phrase, and that it is rightly presented as well.

One caveat is in order: The book is not a history of the subject, and the references should be used as a guide to further reading only, not as a complete picture of the historical development. The attributions are accurate as such, but many important contributions go unmentioned. For instance, in discussing the differentiable sphere theorem (p. 239) the author cites only K. Grove, H. Karcher and E. Ruh's paper—.76 pinching implies standard sphere. This is indeed correct, but the reader is left unaware that Grove-Karcher-Ruh is just a small numerical improvement of the fundamental paper of K. Shiohama and M. Sugimoto (with improvements by Karcher—.87 pinching), and is also left completely in the dark about the earlier history of the theorem, e.g., contributions by Gromoll, E. Calabi and Y. Shikata. This is not a criticism of the book as such—not every book need be a history—but the reader is herewith warned that complete, detailed, historical, even recent historical, references are not provided.

The question arises how this book compares with others that cover the same material. The answer is that there are essentially no others. S. Kobayashi's and K. Nomizu's *Foundations of differential geometry* covers more topics and does more general versions of the basic material, but it does not treat the more advanced topics of the present book in anything like the same depth. J. Cheeger's and D. Ebin's *Comparison theorems in Riemannian geometry* does treat the topics of Chapter 2 (e.g. manifolds of nonnegative curvature and the sphere theorem) of the present book in depth, but its foundational material is cursory and it contains no material along the lines of the present book's Chapter 3 (e.g., geodesics as a dynamical system). In short, the present book presents a unique, and excellent, selection of topics, and it presents them well. In my view, it is the best available introduction to those topics in contemporary Riemannian geometry centering around the geometry of geodesics. No serious student or practitioner of geometry should be without it.

I noted also that, in a world of mathematics books that often seem to be printed on newsprint but priced like first-edition collectors items, this book is a happy exception. It is elegantly designed and printed and costs a quite reasonable $49.00 (for 380 pp.).

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