An interesting question, to which the authors have made substantial contributions, concerns what can be said about the range of a charge. If the range is bounded and infinite, then it is dense in itself; if, in addition, the domain is a σ-field, the range contains a dense sequence of perfect sets. However, the range need not be a Borel set; by applying Kolmogorov's zero-one law and its category analogue, the authors show that on any infinite σ-field there exists a probability charge whose range is not Lebesgue measurable and does not have the property of Baire. Here is one example. Let $\mathscr{F}$ be the field of all subsets of $N = \{1, 2, \dots\}$. For each $A \in \mathscr{F}$ define $\mu_1(A) = \Sigma\{2^{-n}: n \in A\}$, and let μ_0 be a 0-1 valued charge on $\mathscr{F}$ that is equal to zero for every finite set in $\mathscr{F}$. Then $\mu = \frac{1}{2}(\mu_0 + \mu_1)$ is a probability charge on $\mathscr{F}$ whose range has the stated properties. In fact, that part of the range of 2μ that is contained in $(1, 2]$ coincides with a set whose nonmeasurability was proved by Sierpiński [**2**].

References

1. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

2. W. Sierpiński, *Fonctions additives non complètement additives et fonctions non mesurables*, Fund. Math. **30** (1938), 96–99.

3. S. Wagon, *The use of shears to construct paradoxes in* R^2, Proc. Amer. Math. Soc. **85** (1982), 353–359.

JOHN C. OXTOBY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 1, July 1984

0273-0979/84 $1.00 + $.25 per page

A problem seminar, by Donald J. Newman, Problem Books in Mathematics, Springer-Verlag, New York, 1982, 113 pp., $12.95. ISBN 0-3879-0765-3

> There was once a bumper sticker that read, "Remember the good old days when *air* was clean and *sex* was dirty?" Indeed, some of us are old enough to remember not only *those* good old days, but even the days when Math was *fun*(!), not the ponderous THEOREM. PROOF. THEOREM. PROOF, . . . , but the whimsical, "I've got a good problem."
>
> Why did the mood change? What misguided educational philosophy transformed graduate mathematics from a passionate activity to a form of passive scholarship?
>
> In less sentimental terms, why have the graduate schools dropped the Problem Seminar? We therefore offer "A Problem Seminar" to those students who haven't enjoyed the fun and games of problem solving. (Preface to *A problem seminar*).

1. Opening shots. *A problem seminar* is, pound for pound, the finest collection of the problem-solver's art that I have ever read. It is a master class conducted by a man completely in command of his methods. Unfortunately, it is severely compromised by several relatively superficial failings. These failings call to mind the venerable battle in style between the problem-solvers and the theory-builders. The persistence of this struggle without a knock-out blow from either side leads the skeptic to suspect that both fighters are secretly controlled by the same management. I won't waste your time by repeating the usual reasonable and earnest arguments on this subject. We need both the Apollonian theory-building and the Dionysian problem-solving in mathematics. (In today's political climate, I dare not broach the word "d--l-ct-c".)

2. Straw men. Of course, we can also be less reasonable and less earnest: we can deal in stereotypes. Let's consider the *bad* theory-builder and the *bad* problem-solver. The image of the bad theory-builder is essentially the popular anti-intellectual image of the mathematician: a pompous academician working at such a level of abstraction that no one understands him. More specifically, he applies the jargon of Specialty A and the machinery of Specialty B to the questions of Specialty C, hoping fervently that there is nobody in $A \cap B \cap C$ to recognize his vacuity. No theorem is interesting in itself; it must be placed in every possible context and it must lead to an infinite regress of equally uninteresting theorems. He wouldn't recognize a pretty result if it bit him on the axiom. The bad theorem-builder is an intellectual bully and showoff. The projection of mathematics onto his mind contains beauty in its kernel.

The bad problem-solver is less well known outside of mathematics, but his stereotype is just as sharp: he is the mathematical "hacker", a street vendor of isolated facts. He winds them up in the coffee room: "Hey mister! Bet you can't solve this!" Never mind that nobody else cares. His papers exist in a virtual vacuum; there is no context. His proofs favor the flashy *deus ex machina* over the more insightful (but prosaic) explanation. He asks question after question in the hope that someone important will solve one and name it after him. The bad problem-solver is an intellectual bully and showoff. The projection of mathematics onto his mind is a totally disconnected set.

(I should also declare my personal affinity for problem-solving and a self-interest in seeing its promulgation. This orientation arises both from natural inclination and from a pronounced inability to understand Serge Lang's *Algebra* in graduate school. On the other hand, I sleep better knowing that some people *do* understand Serge Lang's *Algebra*.)

3. Brass tacks. *A problem seminar* contains 109 problems and their solutions; most have hints. As might be expected from the author's research interests, the problems are largely redolent of analytic number theory. The author's purpose and style are remarkably well conveyed by his Preface, reproduced here in its entirety.

In this book, Professor Newman is a surgeon at a teaching hospital, performing his delicate operations with great deftness, and explaining to the rapt internes the how's and why's of his procedures. (The how's are proofs; the why's are heuristics.) He leaves the solutions at a stage where the internes can sew up the patient. This book is nominally aimed at advanced undergraduates and beginning graduate students, but anyone of the problem-solving persuasion will find many useful techniques and provocative insights.

The problems are stated in a uniformly elementary and almost uniformly unambiguous manner. The solutions are simultaneously well-motivated and velvety-smooth. More important, the questions are sufficiently interesting that one would *want* to know the answers, even if the solutions were not so slick. Indeed, the author sometimes is almost embarrassed by his own silkiness. Exclamation points abound; the solution to #48 (see below) begins "This is dazzling!". It is. Nobody needs to ask Donald J. Newman why he does mathematics; his joy is palpable. This is Dionysian problem-solving at its finest.

So what am I complaining about? The trouble with *A problem seminar* is that, on the surface, this is a book of *bad* problem-solving. A disastrous decision was made to provide absolutely no documentation for the problems. There is no motivation, no indication of sources, no description of context, no direction for further reading. Further, the book's organization is, charitably, idiosyncratic. Finally, an unwary reader might draw from this book several extremely unfortunate conclusions about mathematics and the process of problem-solving.

The problems in *A problem seminar* appear from nowhere, dazzle us with their solutions and disappear. With a few exceptions (e.g. #1, #48), these are not simply isolated curiosities; there are no matchstick problems. Many problems represent the first outcroppings of a deeper vein of mathematics (e.g. #46, #68). My Illini Hall neighbor, Ken Stolarsky, has reviewed this book for the *Bulletin* of the London Mathematical Society. We spent a morning tossing names back and forth and were able to document, without much effort, about two dozen of the problems. I refer the reader to his review for the particulars (I am sure we just scratched the surface). One source is worthy of isolation: the author was on the Putnam problems committee for the 1972–1974 exams. Five of those problems appear in this book: #17, #20, #45, #86, #96. I am not complaining about the appearance of certain "classics" in this book. Any problem is new to the person who hasn't seen it before and 20% is a very small fraction for a book of this kind. It saddens me that an opportunity was lost to build a bridge between great isolated problem-solving and the larger body of mathematical knowledge. (Hint for the second edition: the pilings and cables are already in place, all you need is the road.)

After passing the Preface, Contents and Format of *A problem seminar* the reader encounters the heading "Problems" followed by fifty-eight problems arranged in a loose set of clumpings. Suddenly there appears "Estimation Theory", with three insightful paragraphs of heuristics, followed by twenty-nine more problems. This pattern recurs six more times, with heuristics for generating functions, limits of integrals, expectations, prime factors, category arguments and convexity separated by twenty-two problems in sets of size one (!) to seven. I found the heuristics extremely valuable. I appreciated the novel allocation of problems by technique rather than the usual algebra, analysis, number theory. But I was baffled by the inconsistency of the arrangement. (Hint for the second edition: double the paragraphs of heuristics and get your organizational act together.)

When we consider *A problem seminar* as a text, rather than as a monograph, these objections are amplified. It is one thing for an analyst to recognize #82 as an uncredited but well-known theorem of Littlewood. It is quite another to spring this as an isolated theorem on an impressionable student just becoming aware of what mathematical research is. The reader might also draw the surely unintended inference that these problems are all new. However desirable it is to convey the passion and fun and games of mathematics, it is also necessary to convey the sense that mathematics is *about* something and has a deep and beautiful structure. Suppose a student solves #96 and becomes fascinated with the question of estimating partial sums of a Taylor series. Where can she go to read a systematic treatment of the subject with more general results? Where can she find out the open questions in the area? If she is lucky, her instructor may be able to help her. But it ought to be down there in black and white: if you want to hold a banquet, you better pay the caterer.

Part of the pleasure of problem-solving is the adventure of the unexpected. You never know in advance whether your problem is soluble, how long the solution ought to be and which techniques you will need. (Otherwise, you are not solving problems, you are working out exercises.) In a book of this kind, the adventure is inevitably tamed into a safer expedition. In *A problem seminar*, almost every solution is one page or less and completely elementary, up to the occasional fact given in advance. I can imagine that this barrage of short brilliant solutions might easily persuade a fledgling mathematician that he has no future as a problem-solver. Part of the "fun and games" of problem solving is polishing a rough messy solution into a presentable sheen. (Hint for the second edition: include at least one "case history" of a problem with its sequence of monotonically nicer solutions.)

Finally, the selection of problems in *A problem seminar* is a bit too narrow for a problem seminar. There is little of what might be called abstract mathematics and no applied mathematics, apart from the probability problems, which are a step below the quality of the rest. No problem involves a matrix or a group or any topology or geometry beyond the plane; complex numbers (let alone functions) appear only once. (Hint for the second edition: Diversify!)

4. Fudge brownies. Every criticism I have presented is easily rectifiable. A problem book stands or falls on its problems; here is a sample. I won't spoil your fun by giving you the solutions. Buy the book and find out yourself. You won't be disappointed.

- #19 Prove that, at any party, two people have the same number of friends present.
- #23 Maximize $2^{-x} + 2^{-1/x}$ over $(0, \infty)$.
- #24 N distinct non-collinear points are given. Prove that they determine at least N distinct lines.
- #46 Let α, β be positive irrationals. Show that the sets $[n\alpha]$ and $[n\beta]$, $n = 1, 2, 3, \ldots$, are complements iff $1/\alpha + 1/\beta = 1$.
- #48 Call an integer square-full if each of its prime factors occurs to the second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

#51 Define x_n by $x_n = x_{n-1} + \frac{1}{2}x_{n-2}$, $x_0 = 0$, $x_1 = 1$. Prove that for $n > 8$, x_n is not an integer.

#68 Find, asymptotically, the number of lattice points in the disc $x^2 + y^2 \leqslant R^2$ as $R \to \infty$.

#73 Given n points in the unit square, there is a shortest curve connecting them. Estimate the longest this curve can be.

#82 Show that if $f(x)$ and $f''(x)$ are bounded, then $f'(x)$ is. (Here $f(x) \in C^2$, and the domain is the whole line.)

#90 Can the positive integers be partitioned into at least two arithmetic progressions such that they all have different common differences?

#96 Show that $1 + n/1! + n^2/2! + \cdots + n^n/n! \sim \frac{1}{2}e^n$.

#109 At each plane lattice point there is placed a positive number in such a way that each is the average of its four nearest neighbors. Show that all the numbers are the same!

BRUCE REZNICK

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 1, July 1984

0273-0979/84 $1.00 + $.25 per page

Clifford analysis, By. F. Brackx, R. Delanghe, and F. Sommen, Research Notes in Mathematics, Vol. 76, Pitman Advanced Publishing Program, Boston, 1982, 308 pp., $19.95. ISBN 0-2730-8535-2

1. Clifford analysis. What is Clifford analysis? The general answer is that it is the development of a function theory for functions which map $\mathbf{R}^n$ into a universal Clifford algebra with a goal being to generalize to this setting properties of holomorphic functions of one complex variable. Other goals are to relate the monogenic functions, the functions which correspond to holomorphic functions in Clifford analysis, to distributions with values in a Clifford algebra and to study the duals of monogenic functions.

In this first section we define universal Clifford algebra and introduce topological and algebraic structures and spaces of test functions and distributions with values in a certain Clifford algebra; although of a rather technical nature, we need these basic definitions and concepts at our disposal in order to be able to compare the Clifford analysis with previous work and to obtain an understanding of Clifford analysis in its generality as presented in the book under review. In subsequent sections we will discuss motivation for the study of Clifford analysis and topics in the analysis, and we will make some conclusions concerning this book.