INTEGRAL REPRESENTATIONS ON HERMITIAN MANIFOLDS: 
THE $\bar{\partial}$-NEUMANN SOLUTION OF THE 
CAUCHY-RIEMANN EQUATIONS

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1. Introduction. Let $D$ be a relatively compact domain in a Hermitian manifold $X$ of complex dimension $n$. The Cauchy-Riemann operator $\bar{\partial}$ extends to a densely defined operator

$$\bar{\partial}: L^2_{0,q}(D) \to L^2_{0,q+1}(D), \quad 0 \leq q \leq n.$$ 

The inner product in $L^2_{0,q}(D)$ is given by

$$(f,g)_D = \int_D f \wedge * g,$$

where $*$ is the Hodge operator defined by the Hermitian structure. If $\bar{\partial}^*$ is the Hilbert space adjoint of $\bar{\partial}$, one defines the complex Laplacian by

$$\Box = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}.$$

Its significance for complex analysis lies in the fact that if $Nf \in \text{dom} \Box$ solves $\Box(Nf) = f$ and $\bar{\partial}f = 0$, then $u = \bar{\partial}^* Nf$ is the unique solution of $\bar{\partial}u = f$ which is orthogonal to $\text{ker} \bar{\partial}$. J. J. Kohn has established existence and regularity properties of the solution operator $N$, giving the solution of minimal norm—the so called $\bar{\partial}$-Neumann operator—in case $D$ is strictly pseudoconvex [5], and in more general cases as well [6]. The proofs are based on a priori estimates in $L^2$-Sobolev spaces, and therefore they do not give any explicit information about the kernels of $N$ or $\bar{\partial}^* N$.

In recent years there has been much interest in finding more explicit and concrete representations of the abstractly defined operators $N$ and $\bar{\partial}^* N$ (see [2, 3, 9, 10, 12]). In [7] we began to study $\bar{\partial}^* N$ by using the calculus of Cauchy-Fantappié kernels in $\mathbb{C}^n$, in analogy to the work of Kerzman and Stein [4] and Ligocka [8] for the Szegő, respectively, Bergman kernel; in contrast to the scalar case, the incompatibility of the Euclidean metric with the complex geometry of the boundary of $D$ turned out to be a major obstruction in the general case.

In the present paper we overcome this obstruction by generalizing the results in [7] to arbitrary Hermitian manifolds; this enables us to then introduce a special Levi metric—similar to the one in [2]—and to establish the required symmetry properties of the kernels. Our main result gives a new and completely explicit integral representation of the principal part of $\bar{\partial}^* N$ on

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range $\partial$. It is likely that these methods will lead to a corresponding representation of $N$ itself (see [11] for the case of the unit ball in $C^n$).

2. Kernels on Hermitian manifolds. Given a Hermitian manifold $(X, ds^2)$ of dimension $n > 1$, we fix a function $\rho$ on $X \times X$ which agrees with the geodesic distance function in a neighborhood $U$ of the diagonal $\Lambda$, and which is positive and $C^\infty$ on $X \times X - \Lambda$. Generic error terms will be denoted by $\mathcal{E}_j$, $j \in \mathbb{Z}$, meaning that $\mathcal{E}_j$ is a double form which is smooth off the diagonal on a region in $X \times X$, and which satisfies $|\mathcal{E}_j| \leq \rho^j$ locally near the diagonal; integral operators with kernels $\mathcal{E}_j$ will also be denoted by $\mathcal{E}_j$.

Define the double form $\mathcal{F}_q$ on $X \times X - \Lambda$ by

$$\mathcal{F}_q = \frac{(n - 2)!(-1)^q}{q!2^{1+q/\pi n}} \frac{(\partial_x \partial_y \rho^2)^q}{\rho^{2n-2}}, \quad 0 \leq q \leq n.$$  

Since $\square = \frac{1}{2} \Delta + \text{terms of order} \leq 1$, standard results in Riemannian geometry (see de Rham [1]) and integration by parts imply the following representation formula (see [7] for the case $X = C^n$).

**Lemma 1.** If $D \subseteq X$ has $C^1$ boundary and $f \in C^1_0(\partial D)$ satisfies the first $\partial$-Neumann boundary condition, then

$$f(y) = -\int_{\partial D} f \wedge * \partial_x \mathcal{F}_q + (\partial f, \partial_x \mathcal{F}_q)_D + (\partial^* f, \partial_y \mathcal{F}_q)_D + (f, \mathcal{E}_{1-2n})_D$$

for $y \in D$.

For a double form $W$ on a subset of $X \times X$, of type $(1,0)$ in $x$ and type $(0,0)$ in $y$, one defines the generalized Cauchy-Fantappié kernels $\Omega_q(W)$ by

$$\Omega_q(W) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \left( \frac{n-1}{q} \right) W \wedge (\partial_x W)^{n-q-1} \wedge (\partial_y W)^q.$$  

It follows that

$$- * \partial_x \mathcal{F}_q = \Omega_q(\partial_x \rho^2 / \rho^2) + \mathcal{E}_{2-2n}.$$  

Now suppose that $D \subseteq X$ is strictly pseudoconvex with $C^\infty$ boundary. Fix a defining function $r \in C^\infty(X)$ for $D$ which is strictly plurisubharmonic in a neighborhood of $\partial D$. By suitably patching functions defined in local coordinates as in the case $D \subseteq C^n$ (see [4 or 7]), one constructs a smooth function $\phi$ on $\overline{D} \times \overline{D}$ with $\phi(x, x) = 0$ for $x \in \partial D$, $\phi(x, y) \neq 0$ if $x \notin \partial D$ or $y \neq x$, $\phi(x, y)$ holomorphic in $y$ for $x$ near $\partial D$ and $\rho(x, y) < \varepsilon$, and $\phi(x, y) = -\phi(y, x) = \xi_3$. If $W = \partial_x \phi / \phi$ and $B = \partial_x \rho^2 [\rho^2 + 2\tau(x)\tau(y)]^{-1}$, let

$$A_q = A_0(W, B) = \frac{1}{(2\pi i)^n} \sum_{\mu=0}^{n-q-2} a_{\mu,q} W \wedge B \wedge (\partial_x W)^\mu \wedge (\partial_x B)^{n-q-\mu-2} \wedge (\partial_y B)^q$$

for $0 \leq q \leq n-2$, and 0 otherwise, with suitably chosen rational constants $a_{\mu,q}$. Set $L_q = (-1)^{q+1} * A_q$ and define $T_q : L^2_{0,q+1}(D) \to L^2_{0,q}(D), 0 \leq q < n$, by

$$T_0g = (g, \partial_x L_0 - * \Omega_0(W) + \partial_x \Gamma_0)_D,$$

$$T_qg = (g, \partial_x L_q - \partial_y L_{q-1} + \partial_x \Gamma_q)_D \quad \text{for } q \geq 1.$$
An analysis of the kernels as in the case $X = \mathbb{C}^n$ shows that $T_q$ is “smoothing of order $1/2$”, that is, $T_q$ is bounded from $L^\infty$ into $\Lambda_{1/2}$.

3. Main results. The Hermitian metric $ds^2$ on $X$ is said to be a Levi metric for $D$ (or rather $r$) if $ds^2$ is conformally equivalent to

$$\sum \left( \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \right) dz_j \otimes d\bar{z}_k$$

in a neighborhood of $bD$.

**Theorem 1.** Let $ds^2$ be a Levi metric for $D$, normalized so that $\|\partial r\|_{ds^2} = 1$ near $bD$. Then $T_q$ is the principal part of $\overline{\partial}^* N$ on the range of $\overline{\partial} : L^2_{0,q}(D) \to L^2_{0,q+1}(D)$.

**Remark.** If $D$ is the unit ball in $\mathbb{C}^n$ with the Euclidean metric, $\sqrt{2}r = 1 - |x|^2$, and $\phi = 1 - (y, x)$, then $T_q \equiv \overline{\partial}^* N$ on the whole space $L^2_{0,q+1}$ (see [11]). It appears likely that the restriction to range $\overline{\partial}$ in Theorem 1 is unnecessary.

Theorem 1 is a consequence of the following fundamental integral representation formula on strictly pseudoconvex domains, valid for arbitrary metrics, and Theorem 3. We denote by $\mathfrak{A}_j$ a generic integral operator whose kernel is admissible of weighted order $\geq j$ (as defined in [7]).

**Theorem 2.** A form $f \in C^1_{0,q}(D) \cap \text{dom} \overline{\partial}^*$ has the representation

$$f = T_q \overline{\partial} f + T^*_{q-1} \overline{\partial}^* f + E_q f$$

$$+ E_{1-2n} f + E_{2-2n}(\overline{\partial} f, \overline{\partial}^* f) + \mathfrak{A}_1 f + \mathfrak{A}_2(\overline{\partial} f, \overline{\partial}^* f),$$

where

$$E_0 f = (f, *\partial x \Omega_0(W))_D$$

and

$$E_q f = (f, \partial x \partial y L_{q-1} - (\partial x \partial y L_{q-1})^*)_D \quad \text{for } q \geq 1.$$

**Theorem 3.** If $ds^2$ is a Levi metric, normalized as in Theorem 1, then $E_q$ is admissible of weighted order $\geq 1$ for all $q \geq 1$.

The proof of Theorem 2 involves Lemma 1 and a generalization of the calculus of Cauchy-Fantappié forms in $\mathbb{C}^n$ to Hermitian manifolds. Theorem 3 is based on a delicate analysis of the leading terms of $\partial x \partial y L_{q-1}$; since these are of weighted order $\geq 0$, but not $\geq 1$ in general, the main point is a cancellation of singularities due to certain symmetries of the kernels. The result holds for arbitrary metrics in case $q = n - 1$, but if $1 \leq q < n - 1$, the Levi metric condition is essential.

We conclude by stating one of the many applications of these results.

**Theorem 4.** Let $ds^2$ be a Levi metric for $D$. For $q \geq 1$ and $f \in L^2_{0,q} \cap \text{dom} \overline{\partial} \cap \text{dom} \overline{\partial}^*$, one has

(i) \[ \|f\|_{\Lambda_{1/2}} \lesssim \|f\|_{L^2} + \|\overline{\partial} f\|_{L^\infty} + \|\overline{\partial}^* f\|_{L^\infty}; \]
and

\[(ii) \quad \|\bar{\partial}^* N f\|_{A^{1/2}} \leq \|f\|_{L^\infty}, \text{ if } f \text{ is } \bar{\partial}-exact.\]

Theorem 4(i) is the analogue in Hölder norms of Kohn's basic estimate. The corresponding version for \(q = 0\) is

\[(i_0) \quad \|f - P_0 f\|_{A^{1/2}} \leq \||\partial f||_{L^\infty}, \]

where \(P_0 : L^2_{0,0} \to L^2_{0,0} \cap O\) is the orthogonal projection. Estimate \((i_0)\) holds for arbitrary metrics; it follows from Theorem 2 and symmetry properties of \(E_0\) (cf. \([7]\)); it was first proved in \([2]\) by different methods for Levi metrics, and in \([8]\) by the above methods for \(X = \mathbb{C}^n\) with the Euclidean metric. Different proofs of Theorem 4 have been announced in \([9\text{ and } 10]\), but, to our knowledge, detailed proofs have not been published.

REFERENCES


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