INTEGRAL REPRESENTATIONS ON HERMITIAN MANIFOLDS: 
THE $\overline{\partial}$-NEUMANN SOLUTION OF THE 
CAUCHY-RIEMANN EQUATIONS$^1$

BY INGO LIEB$^2$ AND R. MICHAEL RANGE

1. Introduction. Let $D$ be a relatively compact domain in a Hermitian manifold $X$ of complex dimension $n$. The Cauchy-Riemann operator $\overline{\partial}$ extends to a densely defined operator

$$\overline{\partial}: L^2_{0,q}(D) \to L^2_{0,q+1}(D), \quad 0 \leq q \leq n.$$ 

The inner product in $L^2_{0,q}(D)$ is given by

$$(f, g)_D = \int_D f \wedge *g,$$

where $*$ is the Hodge operator defined by the Hermitian structure. If $\overline{\partial}^*$ is the Hilbert space adjoint of $\overline{\partial}$, one defines the complex Laplacian by

$$\Box = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \partial \overline{\partial}.$$ 

Its significance for complex analysis lies in the fact that if $Nf \in \text{dom} \Box$ solves $\Box(Nf) = f$ and $\overline{\partial}f = 0$, then $u = \overline{\partial}^* Nf$ is the unique solution of $\overline{\partial}u = f$ which is orthogonal to ker $\overline{\partial}$. J. J. Kohn has established existence and regularity properties of the solution operator $N$, giving the solution of minimal norm—the so called $\overline{\partial}$-Neumann operator—in case $D$ is strictly pseudoconvex [5], and in more general cases as well [6]. The proofs are based on a priori estimates in $L^2$-Sobolev spaces, and therefore they do not give any explicit information about the kernels of $N$ or $\overline{\partial}^* N$.

In recent years there has been much interest in finding more explicit and concrete representations of the abstractly defined operators $N$ and $\overline{\partial}^* N$ (see [2, 3, 9, 10, 12]). In [7] we began to study $\overline{\partial}^* N$ by using the calculus of Cauchy-Fantappié kernels in $\mathbb{C}^n$, in analogy to the work of Kerzman and Stein [4] and Ligocka [8] for the Szegő, respectively, Bergman kernel; in contrast to the scalar case, the incompatibility of the Euclidean metric with the complex geometry of the boundary of $D$ turned out to be a major obstruction in the general case.

In the present paper we overcome this obstruction by generalizing the results in [7] to arbitrary Hermitian manifolds; this enables us to then introduce a special Levi metric—similar to the one in [2]—and to establish the required symmetry properties of the kernels. Our main result gives a new and completely explicit integral representation of the principal part of $\overline{\partial}^* N$ on
range $\overline{\partial}$. It is likely that these methods will lead to a corresponding representation of $N$ itself (see [11] for the case of the unit ball in $\mathbb{C}^n$).

2. Kernels on Hermitian manifolds. Given a Hermitian manifold $(X, ds^2)$ of dimension $n > 1$, we fix a function $\rho$ on $X \times X$ which agrees with the geodesic distance function in a neighborhood $U$ of the diagonal $\Lambda$, and which is positive and $C^\infty$ on $X \times X - \Lambda$. Generic error terms will be denoted by $\mathcal{E}_j$, $j \in \mathbb{Z}$, meaning that $\mathcal{E}_j$ is a double form which is smooth off the diagonal on a region in $X \times X$, and which satisfies $|\mathcal{E}_j| \leq \rho^j$ locally near the diagonal; integral operators with kernels $\mathcal{E}_j$ will also be denoted by $\mathcal{E}_j$.

Define the double form $\Gamma_q$ on $X \times X - \Lambda$ by

$$\Gamma_q = \frac{(n-2)!(-1)^q}{q!2^{1+q}\pi^n} \frac{\partial_x \partial_y \rho^2}{\rho^{2n-2}}, \quad 0 \leq q \leq n.$$

Since $\Box = \frac{1}{2} \Delta + \text{terms of order } \leq 1$, standard results in Riemannian geometry (see de Rham [1]) and integration by parts imply the following representation formula (see [7] for the case $X = \mathbb{C}^n$).

**Lemma 1.** If $D \subseteq X$ has $C^1$ boundary and $f \in C^1_0(\overline{D})$ satisfies the first $\overline{\partial}$-Neumann boundary condition, then

$$f(y) = -\int_{bD} f \wedge \partial_x \overline{\Gamma}_q + (\overline{\partial} f, \overline{\partial} \Gamma_q)_D + (\overline{\partial}^* f, \partial_x \Gamma_q)_D + (f, \mathcal{E}_{1-2n})_D$$

for $y \in D$.

For a double form $W$ on a subset of $X \times X$, of type $(1,0)$ in $x$ and type $(0,0)$ in $y$, one defines the generalized Cauchy-Fantappié kernels $\Omega_q(W)$ by

$$\Omega_q(W) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \left( \frac{n-1}{q} \right) W \wedge (\overline{\partial} W)^{n-q-1} \wedge (\overline{\partial}_y W)^q.$$

It follows that

$$- \partial_x \overline{\Gamma}_q = \Omega_q(\partial_x \rho^2 / \rho^2) + \mathcal{E}_{2-2n}.$$

Now suppose that $D \subseteq X$ is strictly pseudoconvex with $C^\infty$ boundary. Fix a defining function $r \in C^\infty(X)$ for $D$ which is strictly plurisubharmonic in a neighborhood of $bD$. By suitably patching functions defined in local coordinates as in the case $D \subseteq \mathbb{C}^n$ (see [4 or 7]), one constructs a smooth function $\phi$ on $\overline{D} \times \overline{D}$ with $\phi(x,x) = 0$ for $x \in bD$, $\phi(x,y) \neq 0$ if $x \notin bD$ and $y \neq x$, $\phi(x,y)$ holomorphic in $y$ for $x$ near $bD$ and $\rho(x,y) < \varepsilon$, and $\phi(x,y) = -\phi(y,x) = \mathcal{E}_3$. If $W = \partial_x \phi / \phi$ and $B = \partial_x \rho^2 [\rho^2 + 2r(x)r(y)]^{-1}$, let

$$A_q = A_q(W, B) = \frac{1}{(2\pi i)^n} \sum_{\mu=0}^{n-q-2} a_{\mu,q} W \wedge B \wedge (\overline{\partial} W)^\mu \wedge (\overline{\partial}^2 B)^{n-q-\mu-2} \wedge (\overline{\partial}_y B)^q$$

for $0 \leq q \leq n-2$, and 0 otherwise, with suitably chosen rational constants $a_{\mu,q}$. Set $L_q = (-1)^{q+1} A_q$ and define $T_q : L^2_{0,q+1}(D) \rightarrow L^2_{0,q}(D), 0 \leq q < n$, by

$$T_q g = (g, \partial_x L_q - *\overline{\Omega_q}(W) + \overline{\partial} \Gamma_0)_D, \quad T_q g = (g, \partial_x L_q - \partial_y L_{q-1} + \overline{\partial} \Gamma_q)_D \quad \text{for } q \geq 1.$$
An analysis of the kernels as in the case \( X = \mathbb{C}^n \) shows that \( T_q \) is "smoothing of order 1/2", that is, \( T_q \) is bounded from \( L^\infty \) into \( \Lambda_{1/2} \).

3. Main results. The Hermitian metric \( ds^2 \) on \( X \) is said to be a Levi metric for \( D \) (or rather \( r \)) if \( ds^2 \) is conformally equivalent to

\[
\sum \left( \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} \right) dz_j \otimes d\overline{z}_k
\]

in a neighborhood of \( \partial D \).

**Theorem 1.** Let \( ds^2 \) be a Levi metric for \( D \), normalized so that \( \| \partial r \|_{ds^2} = 1 \) near \( \partial D \). Then \( T_q \) is the principal part of \( \overline{\partial}^* N \) on the range of \( \overline{\partial} : L^2_{0,q}(D) \to L^2_{0,q+1}(D) \).

**Remark.** If \( D \) is the unit ball in \( \mathbb{C}^n \) with the Euclidean metric, \( \sqrt{2r} = 1 - |x|^2 \), and \( \phi = 1 - (y, x) \), then \( T_q \equiv \overline{\partial}^* N \) on the whole space \( L^2_{0,q+1} \) (see [11]). It appears likely that the restriction to range \( \overline{\partial} \) in Theorem 1 is unnecessary.

Theorem 1 is a consequence of the following fundamental integral representation formula on strictly pseudoconvex domains, valid for arbitrary metrics, and Theorem 3. We denote by \( \mathfrak{A}_j \) a generic integral operator whose kernel is admissible of weighted order \( > j \) (as defined in [7]).

**Theorem 2.** A form \( f \in C^1_{0,q}(\overline{D}) \cap \text{dom} \overline{\partial}^* \) has the representation

\[
f = T_q \overline{\partial} f + T_{q-1}^* \overline{\partial}^* f + E_q f
+ \varepsilon_{1-2n} f + \varepsilon_{2-2n}(\overline{\partial} f, \overline{\partial}^* f) + \mathfrak{A}_1 f + \mathfrak{A}_2(\overline{\partial} f, \overline{\partial}^* f),
\]

where

\[
E_0 f = (f, * \partial x \Omega_0(W))_D
\]

and

\[
E_q f = (f, \partial x \partial y L_{q-1} - (\partial x \partial y L_{q-1})^*)_D \text{ for } q \geq 1.
\]

**Theorem 3.** If \( ds^2 \) is a Levi metric, normalized as in Theorem 1, then \( E_q \) is admissible of weighted order \( \geq 1 \) for all \( q \geq 1 \).

The proof of Theorem 2 involves Lemma 1 and a generalization of the calculus of Cauchy-Fantappié forms in \( \mathbb{C}^n \) to Hermitian manifolds. Theorem 3 is based on a delicate analysis of the leading terms of \( \partial x \partial y L_{q-1} \); since these are of weighted order \( \geq 0 \), but not \( \geq 1 \) in general, the main point is a cancellation of singularities due to certain symmetries of the kernels. The result holds for arbitrary metrics in case \( q = n - 1 \), but if \( 1 \leq q < n - 1 \), the Levi metric condition is essential.

We conclude by stating one of the many applications of these results.

**Theorem 4.** Let \( ds^2 \) be a Levi metric for \( D \). For \( q \geq 1 \) and \( f \in L^2_{0,q} \cap \text{dom} \overline{\partial} \cap \text{dom} \overline{\partial}^* \), one has

(i) \[
\|f\|_{\Lambda_{1/2}} \preceq \|f\|_{L^2} + \|\overline{\partial} f\|_{L^\infty} + \|\overline{\partial}^* f\|_{L^\infty};
\]
and

\[ \| \partial^* N f \|_{\Lambda^{1/2}} \lesssim \| f \|_{L^\infty}, \quad \text{if } f \text{ is } \bar{\partial}-\text{exact.} \]

Theorem 4(i) is the analogue in Hölder norms of Kohn’s basic estimate. The corresponding version for \( q = 0 \) is

\[ (i_0) \quad \| f - P_0 f \|_{\Lambda^{1/2}} \lesssim \| \partial f \|_{L^\infty}, \]

where \( P_0 : L^2_{0,0} \to L^2_{0,0} \cap \mathcal{O} \) is the orthogonal projection. Estimate \((i_0)\) holds for arbitrary metrics; it follows from Theorem 2 and symmetry properties of \( E_0 \) (cf. [7]); it was first proved in [2] by different methods for Levi metrics, and in [8] by the above methods for \( X = \mathbb{C}^n \) with the Euclidean metric. Different proofs of Theorem 4 have been announced in [9 and 10], but, to our knowledge, detailed proofs have not been published.

REFERENCES


MATHEMATICISCHES INSTITUT, UNIVERSITÄT BONN, 5300 BONN 1, WEST GERMANY

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12222