THE TODA FLOW ON A GENERIC ORBIT IS INTEGRABLE

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ABSTRACT. For the generic orbit of the coadjoint action of the lower triangular group on its dual Lie algebra, we exhibit a complete set of integrals in involution for the associated Toda flow.

The Toda flow [1–3] on an orbit of the coadjoint action of the identity component \( L \) of the lower triangular group on its dual Lie algebra \( \{ S : S = S^T \} \) is generated by the Hamiltonian \( \frac{1}{2} \text{tr} S^2 \). As is well known [1–5], the eigenvalues of \( S \) provide integrals in involution for the flow. In particular, the flow on an orbit consisting of tridiagonal matrices is completely integrable. The purpose of this note is to describe sufficient additional integrals to prove that the Toda flow on a generic coadjoint orbit is also integrable.

For an \( n \times n \) matrix \( M \), define the polynomial

\[
P_k(M, \lambda) \equiv \det((M - \lambda)_{ij} : k + 1 < i \leq n, \ 1 \leq j < n - k),
\]

\[
= \sum_{r=0}^{n-2k} E_{r,k}(M) \lambda^{n-2k-r}, \quad 0 \leq k \leq \left[ \frac{n}{2} \right].
\]

The signs of \( E_{0,k}(S) \) are preserved under the coadjoint action of \( L \). We say that an orbit \( O_S \) through \( S \) is generic if \( E_{0,k}(S) \) is nonzero, \( 0 \leq k \leq \lfloor n/2 \rfloor \).

On a generic orbit define

\[
I_{r,k}(S) = E_{r,k}(S)/E_{0,k}(S), \quad 0 \leq k \leq \lfloor (n-1)/2 \rfloor, \ 1 \leq r \leq n - 2k.
\]

THEOREM 1. The generic orbit \( O_S \) is the set \( \{ U = U^T : \text{sgn} E_{0,k}(U) = \text{sgn} E_{0,k}(S), \ 0 \leq k \leq \lfloor n/2 \rfloor, \ I_{1,k}(U) = I_{1,k}(S), \ 0 \leq k \leq \lfloor (n-1)/2 \rfloor \} \) and has dimension \( 2\lfloor n^2/4 \rfloor \). The functions \( I_{r,k}(S), \ 0 \leq k \leq \lfloor (n-1)/2 \rfloor, \ 2 \leq r \leq n - 2k \), provide \( \lfloor n^2/4 \rfloor \) integrals in involution for the Toda flow. Furthermore, the integrals are independent on a dense open set in \( O_S \).

Let \( \text{GL}_k(n) \), \( 0 \leq k \leq \lfloor (n-1)/2 \rfloor \), denote the identity component of the group of invertible matrices obtained by setting equal to zero all entries which are strictly above the main diagonal and lie in the first \( k \) rows or the last \( k \) columns. A key fact in the proof of Theorem 1 is that each \( I_{r,k} \) is invariant under the restriction to \( \text{GL}_k(n) \) of the coadjoint action of \( \text{GL}(n) \) on its dual Lie algebra, i.e., \( I_{r,k}(g^T M (g^T)^{-1}) = I_{r,k}(M) \) for all \( g \in \text{GL}_k(n) \) and all matrices \( M \). In particular, as \( L \subset \text{GL}_k(n) \), it follows as in the Kostant-Symes theorem (see e.g. [3]) that the Poisson brackets for the \( I_{r,k} \)'s on \( O_S \) can be computed in \( T^*(\text{GL}(n)) \), where the invariance under the full group \( \text{GL}_k(n) \) is then used to prove involution. Also the flows on the orbit \( O_S \) are projections of the associated flows on \( T^*(\text{GL}(n)) \). Indeed, we have

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THEOREM 2 (cf. [3]). For any \( r,k \), let \((Q(t), P(t)), (Q(0), P(0)) = (1, S)\), be the (global) flow induced on \( T^*(\text{GL}(n)) = \{(Q, P): Q \in \text{GL}(n); P \text{ any } n \times n \text{ matrix}\}\) by the Hamiltonian \( I_{r,k}(Q, P) \equiv I_{r,k}(Q^T P) \). Then

\[ S(t) = Z(t)^T S Z(t) \]

solves the \( I_{r,k} \) flow on \( O_S \) with initial condition \( S(0) = S \), where \( Q(t) = Z(t)X(t) \) is the \( QL \)-decomposition of \( Q(t) \).

The flows generated on \( O_S \) by the Hamiltonians \( I_{r,0} \), the symmetric functions, always converge as \( t \to \pm \infty \) [6]. However, the flows generated by \( I_{r,k}, k \geq 1 \), can have periodic orbits. Thus, in contrast to the tridiagonal case, where the invariant sets for the Toda flow are products of lines (see e.g. [5–7]), in the generic case the invariant sets are products of lines and circles.

The \( E_{r,k} \)'s defined above can also be used to prove integrability for the Toda flow on (generic) orbits of band matrices.

Finally, from a different point of view, it is possible to decompose the Toda flow on general orbits into a union of independent Toda flows on traditional (and hence integrable) orbits.

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