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CHARACTERIZING k-DIMENSIONAL UNIVERSAL MENGER COMPACTA

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The disjoint k-cells property (DD^kP), isolated by J. W. Cannon [**Ca**], has played the critical role in the characterization theorems for finite-dimensional manifolds (R. D. Edwards [**Ed**], F. Quinn [**Qu**]) and for manifolds modeled on the Hilbert cube (H. Toruńczyk [**To**]). A metric space X has DD^kP if each pair $f, g\colon I^k \to X$ of maps of a k-cell into X can be arbitrarily closely approximated by maps with disjoint images.

By Toruńczyk's characterization theorem, a compact AR is homeomorphic to the Hilbert cube Q iff it satisfies DD^kP for $k = 0, 1, 2, \ldots$.

On the other hand, the Cantor set $C = \mu^0$ is the only 0-dimensional compactum that satisfies DD^0P (i.e. does not have isolated points). From R. D. Anderson's characterization of the universal curve μ^1 [**An$_2$**] (the 1-dimensional Peano continuum with no local cut points which does not contain a nonempty open set that can be embedded into the plane), it follows that μ^1 is the only connected (C^0), locally connected (LC^0) 1-dimensional compactum that satisfies DD^1P. The construction of the universal curve generalizes to give the k-dimensional universal Menger space μ^k: Subdivide $[0,1]^{2k+1} = A_1$ into 3^{2k+1} congruent $(2k+1)$-cubes, and let A_2 be the union of the cubes adjacent to the k-skeleton of $[0,1]^{2k+1}$. Repeat the construction on each of the remaining cubes to obtain A_3 and, similarly, $A_4, A_5, \ldots$. Then set $\mu^k = \bigcap_{i=1}^{\infty} A_i$.

THEOREM. *If X is a k-dimensional $(k-1)$-connected (C^{k-1}), locally $(k-1)$-connected (LC^{k-1}) compact metric space that satisfies DD^kP, then $X \approx \mu^k$.*

COROLLARY. *Different constructions of the universal k-dimensional space appearing in the literature* (*cf.* [**Mg, Lf, Pa**]) *yield the same space.*

In the proof, a different construction of μ^k is used as a working definition. This construction is more suitable for inductive arguments, since it allows "handlebody decompositions" of μ^k, where each "handle" is a copy of μ^k, the intersection of two "handles" is a copy of μ^{k-1}, the intersection of three "handles" is a copy of μ^{k-2}, etc. This approach leads to a construction of many homeomorphisms $h\colon \mu^k \to \mu^k$ which are used to develop a decomposition theory for μ^k (via Bing's Shrinking Criterion). The main result here is that a UV^{k-1}-surjection $f\colon \mu^k \to X$ is approximable by homeomorphisms provided $\dim X = k$ and X satisfies DD^kP.

The final part of the proof consists of showing that any C^{k-1}, LC^{k-1} metric compactum admits a UV^{k-1}-surjection $f\colon \mu^k \to X$ (a resolving map).

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Omitting compactness and global connectivity, we obtain the characterization of manifolds modeled on μ^k.

THEOREM. *For a locally compact, locally $(k-1)$-connected k-dimensional metric space X, the following statements are equivalent.*

(i) *X satisfies DD^kP.*

(ii) *Each point $x \in X$ admits a neighborhood U homeomorphic to an open subset of μ^k.*

Many theorems from Q-manifold theory translate immediately into μ^k-manifold theory. For example, the Z-set unknotting theorem (properly interpreted) holds for μ^k-manifolds. In particular, μ^k is homogeneous (this fact is well known for $k = 0$, and it was proved by R. D. Anderson [$\mathbf{An_1}$] for $k = 1$). In short, μ^k is "the k-dimensional analogue of the Hilbert cube Q".

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