Lie groups, bundle theory and sheaves, but also is experienced enough to be able, for example, to figure out what kind of induced map is denoted by an asterisk attached to a symbol, or to which of several structures that happen to be lying around a word like "homomorphism" might refer. Even well-prepared readers might have difficulty in deciphering occasional slips into a cryptic style in the places where there is a small error or omission. A bit more redundancy or explanation might have made it possible to guess what was intended by a passage such as the one comprising the first three sentences of Example 1.20 (p. 8), which was among several that I never understood.

It used to be easy to be an expert on foliations. One only had to read Reeb [2] and a few important papers. Now that so much more is known, it is harder, but thanks to Reinhart it is at least possible to get a good idea of the field in a reasonable time. He has written a book reflecting his own tastes rather than some sort of consensus view among foliaters. Even though most will find that some of their favorite topics have been omitted or only given brief mention, I believe that they will agree that the book is more stimulating and interesting because of its individuality. It deserves to be successful and I hope that there will be later editions that will show a little more compulsive attention to detail and pity for the frailties of readers.

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Function theory on planar domains, a second course in complex analysis, by Stephen D. Fisher, John Wiley & Sons, Inc., 1983, xiii + 269 pp., $34.95. ISBN 0-4718-7314-4

The most intensively studied spaces of analytic functions are the Hardy spaces on the unit disk D in the complex plane. For each positive number p, the Hardy space $H^p(D)$ is the set of analytic functions f on the unit disk D such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

The Hardy space $H^\infty(D)$ is the set of bounded analytic functions on D. A sample of the wealth of information that twentieth century mathematicians have discovered about these spaces can be found in the books of Hoffman [4], Duren [2], Koosis [5], and Garnett [3].

Let G be a connected open subset of the complex plane. To avoid technicalities, for the purpose of this review we will also require G to be bounded. To correspond to the case where G is the disk, it is clear that $H^\infty(G)$ should be defined to be the set of bounded analytic functions on G. For $0 < p < \infty$, the correct definition of $H^p(G)$ is not obvious. The space $H^p(D)$ is defined by integrations around circles. Correspondingly, we could take a set of curves in G tending to the boundary of G in an appropriate sense, and consider analytic functions f defined on G such that the set of integrals of $|f|^p$ over these curves is bounded. This procedure leads to an interesting theory, but a more fruitful and natural theory arises from another definition.

In the case of the disk it has been known for over fifty years that a function f analytic on D is in $H^p(D)$ if and only if there is a harmonic function u on D such that $|f(z)|^p \leqslant u(z)$ for every z in D. Now it is clear how to define the Hardy spaces for more general regions G: for any positive number p, a function f analytic on G is said to be in $H^p(G)$ if and only if there is a harmonic function u on G such that $|f(z)|^p \leqslant u(z)$ for every z in G. For regions with smooth boundary, the Hardy space $H^p(G)$ will be the same set of functions as obtained from the definition alluded to in the previous paragraph.

It is easy to verify that the Hardy spaces are conformally invariant. More precisely, if ϕ is an analytic homeomorphism of a region G' onto G, then a function f analytic on G is in $H^p(G)$ if and only if $f \circ \phi$ is in $H^p(G')$. Thus the Riemann Mapping Theorem implies that if G is simply connected, we might as well go back to the unit disk. The main subject of the book under review is the study of Hardy spaces on multiply connected domains. Some theorems from the disk carry over, with the same proofs, to more general regions. Other theorems are true on both the disk and multiply connected regions, but new tools are required for the proof in the more general context. Finally, some entirely new phenomena, which have no analog on the disk, arise on some multiply connected domains.

From the large pool of research done in this area in the last thirty years, Fisher has selected some exciting topics for inclusion in the book. He begins by developing one of the major tools of this subject, the solution of the Dirichlet problem. The Dirichlet problem is said to be solvable on the region G if for every real-valued continuous function U on ∂G, there is a function u continuous on $G \cup \partial G$ such that u is harmonic on G and $u|\partial G = U$. One of the most useful theorems in this subject states that if every connected component of ∂G contains more than one point, then the Dirichlet problem is solvable for G. Once we know that the Dirichlet problem can be solved for the region we are studying, we can imitate many of the results from the disk which depend upon the Poisson kernel.

Fisher devotes a short chapter to the Uniformization Theorem, which Ahlfors has called "perhaps the single most important theorem in the whole theory of analytic functions of one variable" [**1**, p. 136]. This remarkable theorem states that there is an analytic function T from the unit disk D onto our region G such that T is a covering map. This means that for each point w of G, there is an open neighborhood P of w such that each connected component of $T^{-1}(P)$ is mapped homeomorphically by T onto P. Using this covering map T, we can identify $H^p(G)$ with a subspace of $H^p(D)$.

A region whose complement contains only finitely many connected components, none of which is a single point, is conformally equivalent to a region whose boundary consists of finitely many disjoint analytic simple closed curves. It is for such regions that the theory is most developed. Many of the familiar results from the unit disk reappear when G has such a nice boundary. For example, if G is such a region, then every function in $H^p(G)$ has boundary values almost everywhere (with respect to arc length measure) on ∂G. As in the disk, a function in $H^\infty(G)$ is called an inner function on G if its boundary values have absolute value one almost everywhere. Unlike the disk, even for nice G (consider an annulus) it is difficult to write down concrete inner functions. However, one of the pleasant results states that there are plenty of inner functions on G—enough so that their linear span is dense in $H^\infty(G)$.

One crucial property of simply connected regions fails for a multiply connected region G. If u is a real-valued harmonic function on G, then there is not necessarily an analytic function f on G such that $u = \operatorname{Re} f$. Thus the harmonic conjugate of u (which would be $\operatorname{Im} f$ if we could write $u = \operatorname{Re} f$) is not well defined. Fisher (and other authors) often deal with this problem by talking about a harmonic conjugate which is a multiple-valued function with periods. The following theorem can replace this confusing approach and simplify many proofs. It is clear that Fisher (and others) implicitly know this theorem, but it rarely appears explicitly in the literature. Walsh [**6**, pp. 518, 527] indicates a proof using a version of Green's formula. Actually, a simpler proof can be obtained by applying the Cauchy integral formula to the analytic function $u_x - iu_y$.

THEOREM. *Let G be a bounded region of* $\mathbf{C}$ *such that* $\mathbf{C} \sim G$ *has finitely many connected components, which we label $K_0, K_1, \ldots, K_n$, with K_0 the unbounded component. For $1 \leqslant j \leqslant n$, choose a point a_j in K_j. Let u be a real-valued harmonic function on G. Then there exist an analytic function f on G and real numbers $c_1, \ldots, c_n$ such that*

$$u(z) = \operatorname{Re} f(z) + c_1 \log|z - a_1| + \cdots + c_n \log|z - a_n|$$

for every z in G.

Distinguished homomorphisms provide a fascinating example of behavior in an arbitrary region G which has no analog in the disk. Without getting into the details, here is a sample of what can happen. There is a region G which contains the open unit interval $(0, 1)$ and has 1 as an essential boundary point (this means there is a bounded analytic function on G which cannot be extended continuously to $G \cup \{1\}$), but the limit of $f(x)$, as x increases to 1 along the interval $(0, 1)$, exists for every bounded analytic function f on G.

The appearance of a book on a specialized subject tends to inhibit other mathematicians from writing a book on the same topic in the near future, so it is important for the author to produce a book which can serve as a standard reference on the subject. Thus it is a shame that the Bibliography in Fisher's book is not more complete. Fisher's compilation almost ignores the extensive and important work of Soviet mathematicians on the subject under consideration; of the 107 items in the Bibliography, only two refer to work published in the Soviet Union.

The number of minor errors in the book is annoyingly large. Sometimes the proofs have the appearance of having been hastily written and then not carefully checked. Some sections are sufficiently sloppy to cause confusion. In spite of these problems, there was no lack of volunteers to lecture from the book at a year long seminar at Michigan State University. We covered almost the entire book and everyone was enthusiastic about the choice of topics that Fisher made. For anyone interested in complex analysis, there is a lot of fun (and a bit of frustration) to be found in this book.

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Integral representations and residues in multidimensional complex analysis, by L. A. Aĭzenberg and A. P. Yuzhakov, Translations of Mathematical Monographs, Vol. 58, American Mathematical Society, 1983, x + 283 pp., $68.00. ISBN 0-8218-4511-X

One of the most beautiful and, at the same time, useful portions of mathematics is the classical theory of the Cauchy integral: the Cauchy integral theorem and the residue theorem. This theory has found application in remarkably diverse directions, from number theory to hydrodynamics. In addition to such extramural applications, the Cauchy theory has, of course, played an essential role in the development of the magnificent edifice that is the modern theory of functions of a complex variable.

The development of the theory of functions of several complex variables, beginning with Riemann and Weierstrass, has followed rather different lines. In this development the Cauchy integral formula of classical function theory has played an important role, for example, in the usual proof of the Weierstrass preparation theorem. In addition, there is an immediate extension of the Cauchy integral formula, valid for functions holomorphic on polydiscs, and it has important uses, which, in the main, parallel the simpler uses of the one-dimensional theory. However, integral representation formulas of an essentially multidimensional nature have not played a major role in the development of the theory of functions of several complex variables. Thus, the two standard English language texts on several complex variables, Gunning and Rossi's *Analytic functions of several complex variables* [**6**], and Hörmander's *An*