BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

LOCAL MODULI FOR MEROMORPHIC DIFFERENTIAL EQUATIONS

BY DONALD G. BABBITT AND V. S. VARADARAJAN

1. Introduction. This note announces results concerning the parametrization, in the sense of (local) moduli, of the equivalence classes of systems of meromorphic differential equations of the form

$$(*) \qquad du/dz = Au$$

near an irregular singular point (assumed to be $z = 0$). Here u is an n-component column vector, A is an $n \times n$ matrix of meromorphic functions, and equivalence of systems defined by matrices A and B means that there is a meromorphic invertible $n \times n$ matrix x such that

$$(**) \qquad x[A] \stackrel{\text{def}}{=} xAx^{-1} + (dx/dz)x^{-1} = B$$

near $z = 0$. If $\mathcal{F}_{\text{cgt}}$ (resp. $\mathcal{F}$) is the field of quotients of the ring of convergent (resp. formal) power series in z with coefficients in $\mathbf{C}$, $(**)$ defines an action of $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}})$ on $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$, reflecting the fact that $(*)$ goes over to the system $dv/dz = Bv$ under the substitution $v = xu$; replacing $\mathcal{F}_{\text{cgt}}$ by $\mathcal{F}$ leads to the notion of formal equivalence. We note that for any commutative ring R (with unit) equipped with a derivation D, $(**)$ defines an action of $\mathrm{GL}(n, R)$ on $\mathfrak{gl}(n, R)$, with D replacing d/dz; if R is a suitably restricted ring of Laurent series in z with coefficients in the ring of convergent power series in d variables and $D = d/dz$, we obtain the notion of equivalence of analytic families of systems $(*)$ depending on d parameters, which is basic to the theory of local moduli (cf. [**BV2**]).

One parametrizes the equivalence classes of systems $(*)$ in two steps. The first step is the classification up to formal equivalence, i.e., the description of the orbit space $\mathrm{GL}(n, \mathcal{F})\backslash\mathfrak{gl}(n, \mathcal{F})$; the second step is to fix a formal class Ω with $\Omega_{\text{cgt}} \stackrel{\text{def}}{=} \Omega \cap \mathfrak{gl}(n, \mathcal{F}_{\text{cgt}}) \neq \emptyset$, and to classify the systems $(*)$ in Ω_{cgt} up to equivalence, i.e., to describe the orbit space $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}})\backslash\Omega_{\text{cgt}}$. The description of $\mathrm{GL}(n, \mathcal{F})\backslash\mathfrak{gl}(n, \mathcal{F})$; goes back to Hukuhara and Turrittin (see [**BV1**] for extensive references) and is based on the notion of a canonical form. The classical method of studying the second question is based on the technique of Stokes lines and Stokes multipliers [**Bi**, **J**]. Recently this has been examined from a more modern, and essentially cohomological, point of view, notably by Malgrange [**Ma1**, **Ma2**], Sibuya [**S**], and Deligne (cf. [**Be**]). The present note continues this theme by studying the equivalence of analytic families of systems $(*)$ and is based in a fundamental way on the theory of formal equivalence over general rings developed in [**BV2**].

Received by the editors April 4, 1984 and, in revised form, August 15, 1984.
1980 *Mathematics Subject Classification*. Primary 34A20.

For our purposes we define a *canonical form* to be an element of $\mathfrak{gl}(n, \mathcal{F}_{\mathrm{cgt}})$ of the type

$$B = D_{r_1} z^{r_1} + \cdots + D_{r_m} z^{r_m} + z^{-1} C,$$

where (a) $r_1 < r_2 < \cdots < r_m < -1$, the r_i being integers, (b) $C, D_{r_1}, \ldots, D_{r_m}$ are elements of $\mathfrak{gl}(n, \mathbf{C})$ that commute with each other, (c) the D_{r_i} are nonzero and semisimple, and (d) the real parts of all the eigenvalues of C are in $[0, 1)$ (if $m = 0$, $B = z^{-1}C$). For Ω we take the $\mathrm{GL}(n, \mathcal{F})$-orbit of B. We put $\Omega(B) = \Omega_{\mathrm{cgt}}$ and write $X(B)$ for the space $\mathrm{GL}(n, \mathcal{F}_{\mathrm{cgt}})\backslash\Omega(B)$. Our main results show (cf. §3) that $X(B)$ may be viewed in a natural way as a space of the form $G_B\backslash H(B)$, where $H(B)$ is an algebraic variety isomorphic to an affine space $\mathbf{C}^d$ and G_B is an algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$ acting morphically on $H(B)$, and that "local moduli" exist at the "good" points of this quotient space: the restriction to "good" points is essential even in the simplest cases. Our results may thus be viewed as a description of the analytic deformations of the meromorphic differential equations $du/dz = Au$ when one fixes all the formal invariants of the equation, at least when the point of $H(B)$ defined by A is "smooth and stable".

2. The Stokes sheaf St_B and the identification $\mathrm{GL}(n, \mathcal{F}_{\mathrm{cgt}})\backslash\Omega(B) \approx G_B\backslash H^1(\mathrm{St}_B)$. Fix B as in §1 and let $\Psi = \exp\{\sum_{1 \le j \le m} (r_j + 1)^{-1} D_{r_j} z^{r_j + 1}\}$. The *Stokes sheaf* St_B is the sheaf of (in general noncommutative) groups defined on the unit circle T as follows: for any open subset U of T, $\mathrm{St}_B(U)$ is the group of holomorphic maps of the sector $\Gamma(U) = \{z \in \mathbf{C}^\times \mid z|z|^{-1} \in U\}$ into $\mathrm{GL}(n, \mathbf{C})$ such that

$$\Psi g \Psi^{-1} \sim 1 \; (\Gamma(U)),$$
$$dg/dz = z^{-1}[C, g] \quad \text{on } \Gamma(U).$$

Here, the notation ~ 1 $(\Gamma(U))$ in (a) means that, for any closed arc $U' \subset U$ and any $r \ge 1$, we have $\Psi g \Psi^{-1} - 1 = O(|z|^r)$ as $z \to 0$ in $\Gamma(U')$, the O being uniform in $\Gamma(U')$. If U is an arc $\ne T$ and $z_U^C = \exp(\log_U z \cdot C)$, where $\log_U$ is a branch of the logarithm on $\Gamma(U)$, the map $g \to z_U^{-C} g z_U^C$ takes $\mathrm{St}_B(U)$ onto a unipotent algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$ which is independent of the choice of the logarithm. So all the $\mathrm{St}_B(U)$ become unipotent algebraic groups in a natural way. Consequently, if $\mathfrak{U} = (U_i)$ is a finite open covering of T by arcs $\ne T$, the set $C(\mathfrak{U} \colon \mathrm{St}_B) = \prod_i \mathrm{St}_B(U_i)$ becomes a unipotent algebraic group, the set $Z^1(\mathfrak{U} \colon \mathrm{St}_B)$ of Čech 1-cocycles becomes an affine variety on which $C(\mathfrak{U} \colon \mathrm{St}_B)$ acts, and the space of orbits can be naturally identified with $H^1(\mathfrak{U} \colon \mathrm{St}_B)$. As usual, $H^1(\mathrm{St}_B)$ is the union of all the $H^1(\mathfrak{U} \colon \mathrm{St}_B)$ as $\mathfrak{U}$ varies over the coverings as above. If G_B is the centralizer of $C, D_{r_1}, D_{r_2}, \ldots, D_{r_m}$ in $\mathrm{GL}(n, \mathbf{C})$, G_B acts on each $\mathrm{St}_B(U)$ by g, $u \to g[u] = gug^{-1}$, and hence on $H^1(\mathrm{St}_B)$. Our starting point is the following variant of a theorem of Sibuya-Malgrange ([**S**, **Ma1**]; cf. also [**Maj**]).

PROPOSITION 1. *There is a natural map θ from $\Omega(B)$ to $G_B\backslash H^1(\mathrm{St}_B)$ that is constant on the orbits of $\mathrm{GL}(n, \mathcal{F}_{\mathrm{cgt}})$ in $\Omega(B)$ and induces a bijection of $X(B)$ with $G_B\backslash H^1(\mathrm{St}_B)$.*

3. The main theorems. By an analytic family a in $\mathcal{F}_{\mathrm{cgt}}$ we mean a family $\{a(\lambda)\}$ $(\lambda \in \Delta^q)$, where Δ^q is a polydisc in $\mathbf{C}^q$ centered at the origin,

$a(\lambda) \in \mathcal{F}_{\text{cgt}}$ for all $\lambda \in \Delta^q$, and there is an integer $r \geq 1$ such that, for some holomorphic function a' on $\Delta^q \times \{z| \ |z| < \varepsilon\}$, $a(\lambda)$ is the element of $\mathcal{F}_{\text{cgt}}$ defined by $z^{-r}a'(\lambda\colon z)$. This leads in an obvious way to the notion of analytic families in $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$ and in $\text{GL}(n, \mathcal{F}_{\text{cgt}})$. If A and A_1 are analytic families in $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$ defined over Δ^q, they are called *equivalent* if there is an analytic family x in $\text{GL}(n, \mathcal{F}_{\text{cgt}})$ such that $x(\lambda)[A(\lambda)] = A_1(\lambda)$ for all λ in some neighbourhood of the origin. An analytic family A in $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$ is said to be *in* $\Omega(B)$ if $A(\lambda)$ is in $\Omega(B)$ for all λ in some neighbourhood of the origin.

Let Σ be the set of Laurent polynomials $\sigma = \sum_{1 \leq j \leq m} a_j z^{r_j}$, where a_j is any eigenvalue of D_{r_j}, $1 \leq j \leq m$. For $\sigma, \tau \in \Sigma$ with $\sigma \neq \tau$, let $q = q(\sigma, \tau) \leq -2$ be the order of $\sigma - \tau$, c_q the coefficient of z^q in $\sigma - \tau$, and let $S(\sigma, \tau)$ be the (finite) set of rays in $\mathbf{C}^\times$ where $\text{Re}(c_q z^q)$ vanishes. The rays belonging to $\bigcup_{\sigma, \tau \in \Sigma, \sigma \neq \tau} S(\sigma, \tau)$ are called the *Stokes lines* of B. Let $\mathfrak{T}(B)$ denote the collection of all finite coverings $\mathfrak{U} = (U_i)$ of T by open arcs of length $\leq \pi/(|r_1| - 1)$ with the restriction that the ends of the arcs of length equal to $\pi/(|r_1| - 1)$ are not on any Stokes line.

THEOREM 1. (i) $H^1(\text{St}_B)$ *can be given the structure of an algebraic variety which is natural in the following sense*: *for any* $\mathfrak{U} \in \mathfrak{T}(B)$, $C(\mathfrak{U}\colon \text{St}_B)$ *acts freely on* $Z^1(\mathfrak{U}\colon \text{St}_B)$, $H^1(\mathfrak{U}\colon \text{St}_B) = H^1(\text{St}_B)$, *and* $H^1(\text{St}_B)$ *is the geometric quotient of* $Z^1(\mathfrak{U}\colon \text{St}_B)$ *for this action* (*see* [**MF**] *for the notion of geometric quotient*); *moreover, there is a global cross section for this action.*

(ii) $H^1(\text{St}_B)$ *is isomorphic to the affine space* $\mathbf{C}^d$, *where d is the irregularity of B in the sense of Malgrange* (*cf.* [**Be**, *pp.* 233, 238]).

(iii) *The action of* G_B *on* $H^1(\text{St}_B)$ *is algebraic.*

A point $\gamma \in H^1(\text{St}_B)$ is called G_B-*smooth* if there exists a G_B-invariant open set U containing γ such that the geometric quotient $G_B \backslash U$ exists in the category of complex analytic manifolds. Let $H^1(\text{St}_B)^{\text{sm}}$ be the G_B-invariant open set of G_B-smooth points. Let $Y = G_B \backslash H^1(\text{St}_B), \pi$ the natural map $H^1(\text{St}_B) \to Y$, and $Y^{\text{sm}} = \pi(H^1(\text{St}_B)^{\text{sm}})$; Y is given the quotient topology. The sheaf of G_B-invariant analytic functions on $H^1(\text{St}_B)$ defines a sheaf on Y and converts Y into a ringed space; and Y^{sm} is the open subset of points around which this ringed space looks like a complex manifold of dimension $r = d - \delta$, where δ is the maximum dimension of the G_B-orbits in $H^1(\text{St}_B)$.

THEOREM 2. *Fix* $\gamma \in H^1(\text{St}_B)^{\text{sm}}$. *Let A be an analytic family of elements in* $\Omega(B)$ *defined over* Δ^q *such that* $\theta(A(0)) = \pi(\gamma)$. *Then* $\mu(A)\colon \lambda \to \theta(A(\lambda))$ *is an analytic map of a neighbourhood of the origin into a neighbourhood of* $\pi(\gamma)$. *If* A_1 *is another analytic family in* $\Omega(B)$ *defined over* Δ^q *such that* $\mu(A) = \mu(A_1)$ *in a neighbourhood of the origin, then A and* A_1 *are equivalent.*

The proof of this theorem relies heavily on one of the main results of [**BV2**].

THEOREM 3. *Let r be as defined earlier. Then we can find an analytic family in* $\Omega(B)$ *defined over* Δ^r *such that* $\mu(A)$ *is an analytic isomorphism of a neighbourhood of the origin in* Δ^r *with a neighbourhood of the point* $\pi(\gamma)$. *Any such family is universal in the following sense. If* A_1 *is any analytic family in* $\Omega(B)$ *defined over* Δ^q *with* $\theta(A_1(0)) = \pi(\gamma)$, *we can find an analytic map*

$\alpha\colon \Delta'^q \to \Delta'^r$ (primes denote concentric polydiscs) vanishing at the origin such that the families A_1 and $A \circ \alpha$ are equivalent.

If C is semisimple, G_B is reductive, so we are in the paradigm of Mumford [**MF**]. Let us call a point $\gamma \in H^1(\mathrm{St}_B)$ *stable* if its G_B-orbit is closed and has dimension δ, and let $H^1(\mathrm{St}_B)^{\mathrm{s}}$ be the set of stable points; it is G_B-invariant and Zariski open. The statement that $H^1(\mathrm{St}_B)^{\mathrm{s}} \neq \emptyset$ is equivalent to saying that the action of G_B on $H^1(\mathrm{St}_B)$ is generically stable (cf. [**MF**, p. 154]).

THEOREM 4. *Suppose C is semisimple and $H^1(\mathrm{St}_B)^{\mathrm{s}} \neq \emptyset$. Then $Y^{\mathrm{s}} = G_B \backslash H^1(\mathrm{St}_B)^{\mathrm{s}}$ is an irreducible quasi-affine variety of dimension r. If Γ is the set of points γ in $H^1(\mathrm{St}_B)^{\mathrm{s}}$ such that $\pi(\gamma)$ is a simple point in Y^{s}, then $\Gamma \subset H^1(\mathrm{St}_B)^{\mathrm{sm}}$, Γ is dense in $H^1(\mathrm{St}_B)$, and $G_B \backslash \Gamma$ is a complex manifold of dimension r.*

Already in simple examples such as the Bessel and Whittaker equations, nonsmooth and smooth nonstable points exist. In general, $G_B \backslash H^1(\mathrm{St}_B)^{\mathrm{sm}}$ will not be separated. When B is such that the restriction of C to each spectral subspace of $(D_{r_1}, \ldots, D_{r_m})$ has a simple spectrum, then stable points exist, $PG_B = G_B/\mathbf{C}^{\times}$ acts generically freely on $H^1(\mathrm{St}_B)$, and $r = d - n + 1$.

REFERENCES

[**Be**] D. Bertrand, *Travaux récents sur les points singuliers des équations différentielles linéaires,* Sem. Bourbaki 1978/79, Exp. 525–542, Lecture Notes in Math., no. 770, Springer-Verlag, 1980.

[**Bi**] G. D. Birkhoff, *Collected mathematical papers,* Vol. 1, Dover, New York, 1968.

[**BV1**] D. G. Babbitt and V. S. Varadarajan, *Formal reduction theory of meromorphic differential equations: a group theoretic view,* Pacific J. Math. **108** (1983), 1–80.

[**BV2**] ——, *Deformations of nilpotent matrices over rings and reduction of analytic families of meromorphic differential equations,* Mem. Amer. Math. Soc. (to appear).

[**J**] W. Jurkat, *Meromorphe Differentialgleichungen,* Lecture Notes in Math., no. 637, Springer-Verlag, 1978.

[**Ma1**] B. Malgrange, *Remarques sur les équations différentielles à points singuliers irréguliers,* Lecture Notes in Math., no. 712, Springer-Verlag, 1979.

[**Ma2**] ——, *La classification des connexions irréguliers à une variable,* Mathématique et Physique: Sém. École Norm. Sup. 1979–1982, Birhäuser, 1983.

[**Maj**] H. Majima, *Analogues of Cartan's decomposition theorem in asymptotic analysis,* Funkcial. Ekvac. **26** (1983), 131–154.

[**MF**] D. Mumford and J. Fogarty, *Geometric invariant theory,* 2nd Enl. Ed., Springer-Verlag, 1982.

[**S**] Y. Sibuya, *Stokes phenomena,* Bull. Amer. Math. Soc. **83** (1977), 1075–1077.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024