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FINITENESS OF MORDELL–WEIL GROUPS OF GENERIC ABELIAN VARIETIES

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In a series of papers in the 1960s Shimura studied analytic families of abelian varieties with fixed polarization, endomorphism, and level structure. The isomorphism classes of abelian varieties in such a family are in one-to-one correspondence with the points of D/Γ, where D is a symmetric domain and Γ is a discontinuous group of transformations of D. Shimura constructed a fibre system (V, W) where the base V is analytically isomorphic to D/Γ, the fibres are the abelian varieties in the family, and V and W are quasi-projective varieties. The fibre A over the generic point of V is an abelian variety defined over the function field K of V. The main result of this announcement is that, under certain conditions on the endomorphism algebra structure, the group of points of A defined over K is finite. Using completely different techniques, Shioda [**8**] proved this result in the case in which D is the complex upper half-plane and Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

The results in this note are an extension of part of the author's Ph.D. thesis [**9**]. Details will appear elsewhere. I would like to express my sincere thanks to my thesis advisor, Professor Goro Shimura.

1. Let F be an arbitrary totally real number field of degree g over the rational number field $\mathbf{Q}$. Let L be either (a) the field F, (b) a totally indefinite quaternion algebra over F (and view L as embedded in $M_2(\mathbf{R})^g$), or (c) a totally imaginary quadratic extension K of F. Let Φ be a representation of L by complex matrices of degree n so that $\Phi + \overline{\Phi}$ is equivalent to a rational representation of L, and $\Phi(1) = 1_n$ (writing 1_n for the identity matrix of size n). Assume that $[L : \mathbf{Q}]$ divides $2n$, and let $m = 2n/[L : \mathbf{Q}]$. In (c), if $\tau_1, \ldots, \tau_g, \overline{\tau}_1, \ldots, \overline{\tau}_g$ are the distinct embeddings of K in the complex number field $\mathbf{C}$, write r_ν and s_ν, respectively, for the multiplicities of τ_ν and $\overline{\tau}_\nu$ in Φ (then $r_\nu + s_\nu = m$). Suppose $T \in M_m(L)$ satisfies ${}^tT^\rho = -T$, where ^{t} is transpose on $M_m(L)$, and ρ is complex conjugation on K and transpose on each factor of $M_2(\mathbf{R})^g$. In (c), suppose iT^{τ_ν} has the same signature as

$$\begin{pmatrix} 1_{r_\nu} & 0 \\ 0 & -1_{s_\nu} \end{pmatrix}$$

for every ν. Let $\mathcal{M}$ be a lattice in L^m, and let $v_1, \ldots, v_s$ be elements of L^m. Let Ω denote the collection of data $(L, \Phi, \rho, T, \mathcal{M}, v_1, \ldots, v_s)$.

Suppose A is an abelian variety with a polarization C, θ is an embedding of L into $\mathrm{End}(A) \otimes \mathbf{Q}$, and $t_1, \ldots, t_s$ are elements of A of finite order.

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DEFINITION. $(A, C, \theta, t_1, \ldots, t_s)$ *is a* polarized abelian variety of type Ω *if* (1) *there is a holomorphic mapping* ξ *of* $\mathbb{C}^n$ *onto* A *inducing an isomorphism of a complex torus* $\mathbb{C}^n/Y$ *onto* A *satisfying* $\xi(\Phi(a)u) = \theta(a)\xi(u)$ *for every* $u \in \mathbb{C}^n$ *and* $a \in \theta^{-1}(\mathrm{End}(A))$; (2) *if* γ *is the involution of* $\mathrm{End}(A) \otimes \mathbb{Q}$ *determined by* C, *then* $\theta(a)^\gamma = \theta(a^\rho)$ *for every* $a \in L$; (3) *there is an* $\mathbb{R}$*-linear isomorphism* η *of* $(L \otimes_{\mathbb{Q}} \mathbb{R})^m$ *onto* $\mathbb{C}^n$ *such that* $\eta(\mathcal{M}) = Y$, $t_i = \xi(\eta(v_i))$ *for* $i = 1, \ldots, s$, *and* $\eta(ax) = \Phi(a)\eta(x)$ *for every* $a \in L$ *and* $x \in (L \otimes_{\mathbb{Q}} \mathbb{R})^m$; *and* (4) C *determines a Riemann form* R *on* $\mathbb{C}^n/Y$ *such that* $R(\eta(x), \eta(y)) = \mathrm{tr}(xT^t y^\rho)$ *for every* x *and* y *in* $(L \otimes_{\mathbb{Q}} \mathbb{R})^m$.

Write H_r for $\{Z \in M_r(\mathbb{C}) \mid {}^tZ = Z; \mathrm{Im}(Z) \text{ is positive symmetric}\}$ and $H_{r,s}$ for $\{Z \mid \text{complex matrix with } r \text{ rows and } s \text{ columns}; 1 - Z\,{}^t\overline{Z} \text{ is positive hermitian}\}$. Let D be $H^g_{m/2}$ in (a), H^g_m in (b), and $H_{r_1,s_1} \times \cdots \times H_{r_g,s_g}$ in (c). The isomorphism classes of polarized abelian varieties of type Ω are in one-to-one correspondence with the points of D/Γ, where Γ is a suitably defined group of transformations on D (see [**3**] and [**4**]). In [**5**] Shimura showed that for each Ω, one can construct a fibre system $\mathcal{F}$ in which the base V is analytically isomorphic to D/Γ and the fibres are the polarized abelian varieties of type Ω.

THEOREM 1. *If* $\dim(V) \geq 1$ *then the group of points of the generic fibre defined over the function field of* V *is finite.*

The remainder of this paper is a sketch of the proof of Theorem 1.

2. The Mordell-Weil group of Theorem 1 is isomorphic to the group of rationally defined algebraic sections from the base V to the fibre variety W. If V is one-dimensional, one sees easily that these sections extend to global holomorphic sections. For higher dimensions we have the following result, which is a consequence of a result of Igusa (Theorem 6 of [**1**]) when the base variety V is compact.

PROPOSITION. *Let* f *be a rational section from* V *to* W. *Then* f *is defined at every point of* V *so that* f *gives a holomorphic section from* V *to* W.

When $\dim(V) = 1$ and V is compact, the second derivative of a holomorphic section is an automorphic form of weight three with respect to Γ. The Eichler-Shimura cohomology isomorphism (Theorem 8.4 of [**7**]) can be used to show these automorphic forms are zero, and this then restricts the number of holomorphic sections. When D is H^r_1 or $H^r_{1,1}$ with $r > 1$, the use of the Eichler-Shimura cohomology isomorphism is replaced by the application of a theorem of Matsushima and Shimura (Theorem 3.1 of [**2**]), which says there are no automorphic forms of mixed weight with at least one nonpositive weight.

3. The cases of Theorem 1 discussed in §2 can be used to prove the theorem in the remaining cases. We select a large collection of embeddings of base varieties V', for which the theorem is known, into a variety V for which we want to prove the theorem. A section f over V may be pulled back to sections over the varieties V'. Since every section over every V' is of finite order, we can obtain a dense set of points of V which map via f to points of finite order in the fibres over V. To show f is torsion, we must show these orders are bounded. We do this by proving a theorem giving a uniform bound for orders of torsion points on fibres with complex multiplication (Theorem 2 below). The finiteness of the Mordell-Weil group of the generic fibre then follows.

For u in V, write $Q_u = (A_u, C_u, \theta_u, t_1(u), \ldots, t_s(u))$ for the fibre over u. The fibre system $\mathcal{F}$ is defined over a number field k_Ω of finite degree such that for every $u \in V$, $k_\Omega(u)$ is the field of moduli of Q_u (see [**5**]). Call Q_u a "CM-fibre" if A_u is isogenous to $A_1 \times \cdots \times A_t$, where A_i has complex multiplication by a CM-field of degree $2 \cdot \dim(A_i)$, for $i = 1, \ldots, t$ (thus, A_u has CM in the sense of [**6**]).

THEOREM 2. *Let k be any subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ and contains k_Ω. There is a constant B, depending only on the field k and the fibre system $\mathcal{F}$, and independent of the choice of CM-fibre Q_u, so that $|A_u(k(u))_{\text{torsion}}| \leq B$.*

The proof of Theorem 2 requires Shimura's Main Theorem of Complex Multiplication.

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