BOOK REVIEWS

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Intersection theory, by William Fulton, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 2, Springer-Verlag, 1984, xi + 470 pp., $39.00. ISBN 3-5401-2176-5

Introduction to intersection theory in algebraic geometry, by William Fulton, Regional Conference Series in Mathematics, Vol. 54, American Mathematical Society, 1984, v + 82 pp., $16.00. ISBN 0-8218-0704-8

Intersection theory is an old and basic part of algebraic geometry. Algebraic geometry is the mathematics of loci defined by algebraic (polynomial) equations; currently such loci are called schemes. Intersection theory concerns the intersection of two schemes meeting in a third; in other words [7], intersection theory is "the system of assumptions, accepted principles, and rules of procedure devised to analyze, predict, or otherwise explain the nature or behavior of" such an intersection.

The two books under review are complete up-to-date accounts of intersection theory. The Ergebnisse book offers a detailed treatment; the CBMS book, a general introduction. Both present a revolutionary new approach, developed by the author in collaboration with MacPherson, which is technically simpler and cleaner, yet much more refined and general. To better appreciate the subject of intersection theory and the contribution of these books, it is useful to know some history.

Intersection theory was founded in 1720 by Maclaurin, 93 years after Descartes promoted the use of coordinates and equations [**6**, pp. 552–554, 607–608]. Maclaurin stated that two curves, defined by equations of degrees m and n, intersect in mn points. A proof was sought by Euler in 1748 and Cramer in 1750. Finally in 1764 Bezout introduced a more refined method of eliminating one of the two variables from the two equations, producing a polynomial in one variable of minimal degree, which he proved is equal to mn. (Euler did so independently the same year.) Bezout went on to treat the case of r equations in r unknowns [**1**, p. 292], and so most any theorem of intersection theory about projective r-space is called Bezout's theorem.

Intersection theory remained centered around Bezout's theorem for a century and a half. Maclaurin used it to show that an irreducible curve of degree n has at most $(n-1)(n-2)/2$ singular points. Maclaurin and others used it to determine the degree of geometrically defined loci. Goudin and du Sejour in

1756 [**6**, p. 554] found that a curve of degree n can have at most $n(n-1)$ tangents with a given direction. Poncelet in 1822 and Plucker in 1834 determined how much the number $n(n-1)$ has to be diminished when the curve has nodes and cusps. Salmon in 1847, generalizing this work to surfaces that are singular along a curve, ran into a new problem—the problem of excess intersection—which he proceeded to solve; he had to determine the number of isolated points in the intersection of three surfaces that pass through a common curve. Steiner, in 1848, used Bezout's theorem implicitly and Bischoff, in 1859, used it explicitly to find the number of conics tangent to five given ones [**4**]. They correctly found that the six coefficients of the equation of a conic tangent to a given one satisfy a homogeneous equation of degree 6, but they incorrectly concluded that the number was 6^5, or 7776. Cremona, in 1864, pointed out their error: every double line appears tangent to each of the five conics, but there are infinitely many double lines, so the number 6^5 has no enumerative significance. In many questions it is necessary to determine the multiplicity of appearance of a point in an intersection. Theoretical work on this issue was undertaken by Zeuthen in 1872 and Halphen in 1874, who used infinitesimals [**9**, p. 288], and by Smith in 1874, who used the resultant [**1**, pp. 294–295].

A revolutionary change in intersection theory took place in 1879 with the appearance of Hermann Schubert's book, *Kalkül der abzählenden Geometrie*. Schubert introduced explicitly the first intersection rings and, implicitly the operations of pullback and pushout. He applied these tools to enumerative geometry with great success, systematized and simplified much earlier work, and solved problems that had previously defied attack [**5**].

Schubert's work grew out of Chasles's [**4**]. In 1864 Chasles gave the first theory of enumerative geometry. In particular, he found many valid formulas and correct numbers, including the correct number of conics tangent to five others, 3264. In 1873 Halphen observed that Chasles's expression for the number of conics satisfying five conditions could be factored formally into the product of the modules of the conditions. The module of a condition, a notion introduced by Chasles, is an expression, $\underline{a}m + \underline{b}n$, where $\underline{a}$ and $\underline{b}$ are integers depending only on the condition, and m and n are variables. When m is set equal to the number of conics in a given 1-parameter system that pass through a general point, and n to the number tangent to a general line, then the module yields the number of conics satisfying the condition. Chasles's expression for the number of conics satisfying five conditions is obtained by formally expanding the product of the five modules and then replacing each term $m^i n^{5-i}$ by the number of conics that pass through i points and are tangent to $5-i$ lines.

Schubert's brilliant leading idea is the following. Represent geometric conditions by algebraic symbols. Given two independent conditions with symbols x and y, represent the condition that either one or the other holds by the sum $x+y$, and represent the condition that both hold simultaneously by the product xy. Consider the symbols x and y to be equal if the conditions are equivalent for enumerative purposes; that is, they are satisfied by the same numbers of figures in an arbitrary system, provided only that both numbers are finite. Then the symbols form a commutative ring of functionals on the systems. For example, the module of a condition on conics, $\underline{a}m + \underline{b}n$, may be interpreted as a symbolic expression representing the given condition as a linear combination of two symbols m and n, which in turn represent the elementary conditions of passing through a point and being tangent to a line. In other words, the original condition is equivalent to the alternative condition of passing through one of $\underline{a}$ points or being tangent to one of $\underline{b}$ lines. Chasles's hard-won expression for the number of conics satisfying five conditions is now obvious!

Schubert considered what he called the problem of characteristics to be the main theoretical problem of enumerative geometry. The problem is to find, for a given sort of figure, a finite number of basic conditions in terms of which an arbitrary condition can be expressed. For example, in Chasles's theory a condition on conics is expressed as a linear combination of the two elementary conditions, namely, to pass through a point and to be tangent to a line. Schubert solved the problem of characteristics in a number of cases via an appropriate coincidence formula. A coincidence formula is a formula for the condition that two figures varying independently should coincide.

Schubert was not fully satisfied with the mathematical foundation available for his theory, but he forged ahead anyway. Philosophically geometry was still viewed as a natural science. Schubert based his work on the two great 19th century principles of geometry, the Chasles correspondence principle and the principle of conservation of number (or law of continuity). The latter has been traced back 200 years to Kepler and Leibniz and has had a stormy history [**6**, pp. 385–387, 841, 843–845]; [**3**]; (Ergebnisse, pp. 193–4). Schubert thought of it as saying that a continuous variation in the figures defining a condition will not change the number of figures satisfying the condition. The correspondence principle is of lesser mathematical depth and consequence; for more information see [**8**, **5**] and the Ergebnisse book. Hilbert predicted that providing a rigorous treatment of Schubert's work would be one of the great projects of 20th century mathematics [**3**]; he proposed it as his 15th problem.

Thanks primarily to Severi (1912, 1916) and van der Waerden (1930), Schubert's ideas are now usually expressed in the following terms: a parameter scheme, whose points represent the figures; subschemes, whose points represent the figures in the various systems; cycles (formal linear combinations of reduced and irreducible subschemes) whose "points" represent the figures satisfying the various conditions; the intersection product of two cycles, which corresponds to the symbolic product of two conditions; numerical equivalence of cycles, which corresponds to enumerative equivalence of conditions; the theorem that algebraic (or continuous) equivalence implies numerical equivalence, which corresponds to the principle of conservation of number; a Kunneth decomposition of the diagonal, which corresponds to a coincidence formula; the method of reduction to the diagonal, which is the basis of Schubert's method of solving the problem of characteristics given the corresponding coincidence formula; the (deep) theorem of finite generation of the cycles modulo numerical equivalence, which implies the existence of a solution to the problem of characteristics in every case.

Today the term "Schubert calculus" is often used in a limited fashion. This is understandable but unjustified. The term is used simply to honor Schubert's solution in 1885–1886 of the problem of characteristics for linear spaces of arbitrary dimension; in other words, to honor his determination of the natural basis of the cycle classes on the Grassmannian. Complemented by the work on the multiplicative structure of the intersection ring done by Schubert himself, Pieri in 1893–1895, and Giambelli in 1903, this work has been particularly significant and inspiring (Ergebnisse Chapter 14; CBMS Chapter 6).

A new source of motivation for work on the foundations of algebraic geometry, in general, and on intersection theory, in particular, appeared in 1940 when Weil announced that he had solved two outstanding problems in number theory; he proved the Riemann hypothesis for fields of algebraic functions in one variable over a finite field of constants, and he proved that Artin's nonabelian L-functions over such fields are polynomials. However, Weil's proofs depended on an analogue of the theory of correspondences; specifically, he needed the form of the theory developed by Severi in 1926, but he needed it in positive characteristic. There followed an intense period of activity, lasting about fifteen years, led by Weil himself. The upshot was a new foundation for algebraic geometry, including an algebro-geometric calculus of cycles based on a local theory of intersection multiplicities, all valid in any characteristic.

The intersection ring on a nonsingular scheme has for half a century been constructed in two separate steps (Ergebnisse, p. 151): (1) developing a calculus of cycles and (2) moving one of two given cycles into general position so that the intersection cycle becomes well defined. The motion is usually parametrized by the projective line, or by a series of projective lines, and then the original cycle and its replacement are said to be rationally equivalent. A complete rigorous treatment from scratch along these lines, albeit in characteristic 0, was given in 1952 by Hodge and Pedoe [**2**]. The 1958 Chevalley seminar focused on rational equivalence, and the ring of rational equivalence classes was named the Chow ring, in honor of Chow's work in 1956. It would certainly be unfortunate, as the author says (Ergebnisse, p. 16; CBMS, pp. 40–41), if, because of a name, the importance of the contributions of Severi and his followers (e.g., B. Segre, Todd, Zariski, Samuel, Grothendieck, etc.) is forgotten. It would be equally unfortunate to forget that the original idea of an intersection ring is Schubert's.

The third major source of motivation for work in intersection theory is the Riemann-Roch theorem. The original theorem, which appeared in 1864, concerns a smooth algebraic curve or compact Riemann surface; it gives a formula for the number of linearly independent rational functions that have poles at worst of prescribed orders at prescribed points. The theorem has attracted a great deal of attention and has been greatly generalized. In the process much ancillary mathematics of independent interest has been developed, including sheaf cohomology, Chern classes and K-theory. The theorem is the subject of two chapters of the Ergebnisse book and one of the CBMS book.

The revolutionary approach of the books under review has two major aspects: a suggestive point of view and a flexible technical device. The point of view is that intersection theory is primarily concerned with the construction and study of operators on cycle classes on singular schemes; the technical device will be discussed below. There is no need for a preliminary theory of intersection multiplicities nor for a general moving lemma. Indeed, a theory of multiplicities is a consequence of the basic constructions. Furthermore, because the moving lemma is unnecessary, the schemes involved do not have to be quasi-projective, and the theory does not have to be set over a field. This point of view of operators actually comes closer to Schubert's original one than does the traditional point of view.

The fundamental operation on classes is now pullback along a regular embedding ("regular" means that the subscheme is cut out locally by a sequence of functions whose Koszul complex is exact). For example, the diagonal map of a smooth scheme is a regular embedding, and the intersection product of two cycles is now defined as the pullback of their cartesian product along the diagonal embedding. In general, the pullback is constructed by degenerating the ambient scheme into the normal cone of the subscheme (a bundle, in the case at hand). Correspondingly, a representative cycle is degenerated into a cycle on the cone. (In fact, if the cycle is the cycle of a scheme, then it degenerates into the cycle of the normal cone in this scheme of the scheme-theoretic intersection with the regularly embedded subscheme.) Thus the situation is reduced to the far simpler case in which the embedding is the zero section of a vector bundle. In this case it is easy to move the cycle so that it meets the zero section properly. This device of reduction to the normal cone has had an interesting history (Ergebnisse, pp. 90–91). In particular, Verdier, in 1974–1975, introduced the operation of pullback along a regular embedding of arbitrary dimension and constructed it by reducing to the normal cone; although he worked within traditional intersection theory, nevertheless he took a key step forward.

The device of reduction to the normal cone leads to a number of refinements, whose development occupies a large portion of both books. These are the following. The intersection class is, by construction, a well-defined class on the set-theoretic intersection, not just on the ambient scheme. If the intersection is proper (that is, the intersection is of minimal dimension), then each component of the set-theoretic intersection appears in the intersection class with a certain multiplicity, which is easily seen to be equal to the multiplicity that is assigned by other theories. When the subscheme is deformed in a system of regular embeddings and simultaneously the cycle is deformed, the pullback can be easily compared to the limit of the pullbacks, because the normal bundle is a first-order approximation to the ambient scheme, somewhat akin to the topologists' tubular neighborhood. It is not hard to see that, when the normal bundle is positive, so is the pullback. Degenerating instead to the completion of the normal bundle at infinity, expressing the image of the zero section in terms of the Chern classes of the normal bundle and the tautological class, operating with this expression on the cycle, and pushing the result down, we get a lovely formula for the pullback. This formula in turn yields the residual-intersection formula, the double-point formula, the excess-intersection formula, and Grothendieck's key formula.

For the device of reduction to the normal cone to work, it is not necessary that the class being pulled back be supported on the ambient scheme. It may be supported on any scheme mapping into the ambient scheme. Then the pullback is supported on the fibered product. This refinement is very important, since many questions may be reduced to the case of divisors by blowing up. The formal properties of this refined pullback and other operators (flat pullback, proper pushout, Chern operators, etc.) are most conveniently expressed in the bivariant language developed in Chapter 17 of the Ergebnisse book and §10.3 of the CBMS book.

In the Ergebnisse book intersection theory is developed in detail and from scratch. Moreover, it is developed with remarkably few technical prerequisites from algebra and algebraic geometry. A basic first-year introduction to the theory of schemes through sheaf cohomology should be adequate preparation. Nevertheless, it may be necessary from time to time to fill in some additional background material. To facilitate the job there are two appendices and many references. In sum, the book makes an excellent choice of text for a second-graduate level course or for an advanced student seminar.

The first six chapters of the Ergebnisse book contain the basic constructions and main theorems. The remaining fourteen chapters depend on these six, but not really on each other. These fourteen present additional general theory and many applications. Nearly half the material in the book goes under the label of example. Most of it is concrete and illustrates the theory. However, the word "example" is also used in the sense of corollary, and it may indicate a further, but more or less straightforward, development of the theory. Each chapter begins with a summary of its contents and ends with a guide to the literature and a number of historical notes, making the book more readable and useful. Doubtless, the book will become a standard reference.

The books under review are destined to go through many editions. Therefore, each generation of readers will serve the next by providing the author with a list of errata and comments. The books are well written and may be recommended to anyone interested in algebraic geometry. The mathematical community owes the author a great debt of graditude for these wonderful books.

References

1. L. Berzolari, *Theorie générale des courbes planes algébriques*, Encyclopedie des Sciences Mathematiques, III.3, Gauthier-Villars, Paris, 1915.

2. W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry*, Vol. 2, Cambridge Univ. Press, 1952.

3. S. L. Kleiman, *Problem* 15. *Rigorous foundation of Schubert's enumerative calculus*, Mathematical Developments Arising from Hilbert Problems, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R. I., 1976, pp. 445–482.

4. ______, *Chasles's enumerative theory of conics: a historical introduction*, Studies in Algebraic Geometry, Studies in Math., 20, Math. Assoc. Amer., Washington, D. C., 1980, pp. 117–138.

5. ______, *An introduction to the reprint edition, of Kalkül der abzählenden Geometrie* by H. Schubert, 1879; reprinted by Springer-Verlag, 1979.

6. M. Kline, *Mathematical thought from ancient to modern times*, Oxford Univ. Press, 1972.

7. W. Morris (Editor), *The American heritage dictionary of the English language*, American Heritage, New York and Houghton Mifflin, Boston, 1971.

8. C. Segre, *Intorno alla storia del principio di correspondenza e dei sistemi di curve*, Bibl. Math. **6** (1982), 33–47 = *Opere*, Vol. 1, Edizioni Cremonese, Rome, 1957, pp. 185–197.

9. H. G. Zeuthen and M. Pieri, *Géomètrie enumerative*, Encyclopédie des Sciences Mathématiques, III.4, Gauthier-Villars, Paris, 1915.

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Mathematical experiments on the computer, Academic Press, New York, 1982, xvii + 525 pp., $39.50. ISBN 0-1230-1750-5

Mathematics is ancient, the computer is new. Both involve the expenditure of energy to manipulate symbols and thereby create structure. At the very least they complement each other, though they often become corrupting influences. Hilbert hoped that mathematics could be managed by computation; Godel proved that it couldn't. In some quarters limitations on the management of computation, e.g., program verification, by mathematics is taken as a measure of the immaturity of our understanding of computation. Rejecting those august heights of mutual absorption, we find that a pragmatic use of the one by the other is healthy for both.

This book deals with one such use: the computer as a calculator to lend credence to conjectures that could be fragments of theorems or suggestive of theorems. The conjectures have in common a significant computational component. None involves the use of the computer as primarily a manipulator of formuale or logical truths. The author is careful to point out that he is considering experiments that are driven by numerical computation. In a more perfect world an experiment on the computer could invoke both the formula manipulation world exemplified by MACSYMA [**1**] and that recommended by the author. Unfortunately the world of the former speaks LISP [**2**] and the latter speaks APL [**3**] and the two in each other's presence are tongue-tied. Some day....

Conjectures of any kind tend to be plastic and tentative. The programming language and computing environment used in exploration should have the same behavior—programs should be easy to write, test, generalize, and discard. Even more, they should be terse and support a great deal of implicit logical and mathematical structure that would otherwise need to be made explicit by programming. The author's choice of APL is to be applauded. The APL environment, its workspace and library, the terseness of its programs, and the ease with which APL programs can be tested and modified make it the best available language for the purposes Grenander has in mind. Unfortunately there is a viper in this Garden of Eden: Learning to program well in APL is considerably more difficult than in any other language. Even though the mastery pays big dividends, most casual users are unwilling to make the required initial time investment. Other programming languages such as BASIC