and testing becomes largely symbolic. More need by mathematicians for a mixed APL LISP environment may push computer science to develop a good one. In the world of systems, traffic, not theory, promotes development.

Since a significant part of mathematics education deals with conjecture and proof, this book suggests how the computer could play an important part. For example, the study of calculus, both elementary and advanced, would benefit enormously from the inclusion of an experimental component that goes far beyond the usual elementary numerical analysis applications. Grenander is implying that with skill in programming and use of APL, the experiments can be significant and the programming labor need not dominate the effort required to master either the art or mechanics of mathematics.

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Chain conditions in topology, by W. W. Comfort and S. Negrepontis, Cambridge Tracts in Mathematics, Vol. 79, Cambridge University Press, New York, 1982, xiii + 300 pp., $39.50.

Set theory and topology have been bedfellows for a long time. Hausdorff's classic text *Mengenlehre* [**2**], for example, devotes only four chapters to set theory; the remaining six, which comprise three-quarters of the book, deal with point-set topology, especially the theory of metric spaces. Perhaps a better translation of the title would be *The theory of point sets*. A similar approach is found in Kuratowski's book [**3**], except that he devotes even less space to set theory, and he has the decency to entitle the book *Topologie*. And while these books were being composed, Sierpinski was gathering the material, largely topological, which would make up his book [**5**] on the continuous hypothesis.

More recently, after a period during which the two subjects developed separately, there has been a dramatic *rapprochement*, the unifying factors being Cohen's discovery of forcing and the subsequent explosion of work in set theory. Consider, for example, the history of the normal Moore space conjecture, which asserts that every normal Moore space is metrizable. (See M. E. Rudin's monograph [**4**] for an account of all but the most recent parts of this story.) First F. B. Jones showed that if $2^{\aleph_0} < 2^{\aleph_1}$, then every *separable* normal Moore space is metrizable. Then Bing showed that if there is Q-set, i.e., an uncountable set of real numbers every subset of which is a relative F_σ, then there is a nonmetrizable separable normal Moore space, and later Silver deduced from Martin's Axiom (see below), which had recently been shown consistent, that Q-sets exist. Next Peter Nyikos used a strong axiom about the extension of product measures, which he called PMEA, to deduce the full normal Moore space conjecture, and Kunen showed that relative to the existence of a very large cardinal, namely one which is strongly compact and, hence, at least measurable, PMEA is consistent. Finally, Fleissner showed that if the continuum hypothesis holds or if there are no inner models of set theory with many measurable cardinals, then the normal Moore space conjecture is false.

Here, then is a thoroughly topological problem that involves most of the aspects of modern set theory: forcing, constructibility, and large cardinals. It seems very likely that the "solution" of this and many other old topological problems is not quite what the original formulators had in mind. The Moore space problem also illustrates the peculiar sociology of work in set-theoretic topology, as it is called nowadays. It turns out that set-theoretic intuition and topological intuition have not been passed out equally, and most mathematicians incline strongly toward one side or the other. Only a privileged few are truly ambidextrous. This tends to result in the formation of symbiotic pairs, in which a topologist will formulate and translate topological problems for a set theorist to attack. The idea of thus doing "applied" set theory holds great charm for set theorists, who have always regarded themselves as the purest of pure mathematicians.

But it is not just a question of basing topology on set theory. It turns out that many important set-theoretic ideas, including forcing itself, can be expressed topologically. The connection runs as follows. Forcing is usually done with respect to a partial ordering P where the elements of P, called *conditions*, are thought of as conveying partial information about a new universe of set theory being constructed. If $p \leqslant q$ then let us say that p *extends* q. Now every partial ordering is equivalent, for forcing purposes, to a unique complete Boolean algebra in which it is embeddable (hence the counterintuitive definition of "extends" as "$\leqslant$"), and the Boolean algebra may be studied topologically in terms of its Stone space. But the connection goes farther than that. Recall that an open set in a topological space is *regular open* iff it is the interior of its closure; intuitively, it has no cracks in it. It is a straightforward exercise to see that the regular open sets in X form a complete Boolean algebra $\mathrm{RO}(X)$, which may now be used for forcing purposes. In fact, if P is a partial ordering which is given a topology by declaring $T_p = \{q \in P\colon\ q \leqslant p\}$ open for all $p \in P$, then $\mathrm{RO}(P)$ is the complete Boolean algebra equivalent to P mentioned above. One can also compute the Stone space of $\mathrm{RO}(X)$, which is called the *Gleason space* $G(X)$ of X, and which can be substituted for X for many purposes. In general, $G(X)$ is not homeomorphic to X since, for example, $G(X)$ is always compact and zero dimensional.

For a set theorist the nicest property a partial ordering can have is the *countable chain condition*. If $p,q \in P$ then we can think of the information embodied in p and q as being inconsistent, and we say p and q are *incompatible*, if no r in P extends both p and q. P has the countable chain condition (the c.c.c) if every pairwise incompatible subset of P is countable. Intuitively, there cannot be very much disagreement within P. Such partial orderings are important partly because forcing with them cannot destroy existing cardinal numbers

When this idea is translated into topology, one arrives at the notion of a c.c.c. space, which is taken to be a (regular) space with the property that every pairwise disjoint system of open sets is countable. Martin's Axiom, which is usually phrased as an assertion about c.c.c. partial orderings, becomes the statement that no (infinite) compact c.c.c. space is the union of fewer than $2^{\aleph_0}$ nowhere dense sets, and thus has a clear meaning as a kind of strong Baire category theorem.

Since mathematicians will never leave a good idea alone, there has naturally been a lot of work on variants of the c.c.c. For example, it is not guaranteed that the product of two spaces (or partial orderings) with the c.c.c. has the c.c.c. This is true under Martin's Axiom if $2^{\aleph_0} > \aleph_1$, but false if the continuum hypothesis holds (due to Laver and to Galvin (with an easier proof)). On the other hand, if we make a stronger requirement on the spaces, we may recover the product theorem. Say X has *caliber* (*precaliber*) $\aleph_1$ if every uncountable collection of open sets has an uncountable subcollection with nonempty intersection (with all finite intersections nonempty). Then these notions are preserved under the formation of products, and they provide the stepping-off point for the book of Comfort and Negrepontis, *Chain conditions in topology*.

The authors expect the reader to come to the book prepared, so they do not offer much motivation for their definitions. It may be worth remarking, therefore, that the first author has written a very nice introduction, specifically for this book, which, unfortunately, appears elsewhere [**1**]. In particular, he defines there what he means by a chain condition on a space as "roughly speaking, a condition concerning one or more cadinal numbers and intersections of systems of open sets". This is a pretty broad interpretation, but it is still fundamentally topological, and the set theorist will find several species of chain conditions not treated here.

Nevertheless, much material is packed into this small book. In addition to calibers and precalibers, the authors treat compact-calibers and pseudo-compactness. A space X has *compact-caliber* α if given α nonempty sets there is a compact set in X which meets α of them, and X is *pseudo-α-compact* if given α nonempty sets there is a point x in X such that every neighborhood of x meets α of the open sets. In fact, these notions are greatly generalized; the authors define caliber (α,β,γ), compact-caliber (α,β,γ), and so forth, and when they consider preservation under products they treat box products of various sizes as well as the usual product.

In addition to results of Argyros, Comfort, Negrepontis, Tsarpalias and other topologists, the authors include set-theoretical arguments due to the likes of Galvin, Hajnal, Laver and Shelah. Although most of the results have already been published, they are scattered through many journals, and are brought together here for the first time. This book will be a valuable source for topologist and set theorist alike, although set theorists may experience a little frustration with it. The lack of motivation is an annoyance, there is no discussion of independence results except in notes at the ends of several chapters, and there are some minor notational irregularities, such as referring to δ-systems as "quasi-disjoint families". But this is hardly serious and is due simply to the fact that the authors are topologists.

As a set theorist, this reviewer has naturally required the assistance of a topologist during the preparation of this review and would here like to thank Frank Tall for some helpful discussions. All opinions expressed here, however, are due to the reviewer.

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Linear and combinatorial optimization in ordered algebraic structures, by U. Zimmermann, Annals of Discrete Mathematics, vol. 10, North-Holland Publishing Company, Amsterdam, 1981, x + 380 pp., $61.00, Dfl. 125.00. ISBN 0-4448-6153-X

Many optimization problems arise in connection with systems which incorporate discrete structures for which the mathematics is combinatorial rather than continuous: one thinks of sequencing, scheduling and flow-problems and of the great variety of questions which can be reformulated as path-finding, circuit-finding or subgraph-finding problems on an abstract graph. To match the growing interest in such problems arising from, for example, operations research and systems theory, the past thirty years have witnessed a vigorous growth in the theory and practice of combinatorial optimization.

A related, but perhaps less well-known, development has been in the application of ordered algebraic structures to optimization problems. This application is made relevant by the fact that many optimization questions depend essentially on the presence of two features: an algebraic language within which a system can be modelled and an algorithm articulated; and an ordering among the elements which enables a significance to be given to the concept of minimization or maximization. A familiar example here is a well-known method of resolving degeneracy in linear programming which depends upon the fact that the simplex algorithm may be extended to linear programs in which values are taken in a certain ordered ring.

By adopting this algebraic point of view we can make useful reformulations: certain bottleneck problems become algebraic linear programs; certain