no-nonsense style of a research monograph, this book provides a rewarding look at some of the recent work of the Soviet school of complex analysis in several variables for those with some previous experience in the subject.

REFERENCES

1. J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. (2) **81** (1965), 451–472.

2. B. M. Weinstock, *Continuous boundary values of analytic functions of several complex variables*, Proc. Amer. Math. Soc. **21** (1969), 463–466.

G. B. FOLLAND

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Univalent functions, by Peter L. Duren, Grundlehren der mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983, $46.00, xiv + 382 pp. ISBN 0-3879-0795-5

Prefatory Note (added August, 1984). After the review appearing below was submitted I learned that Bieberbach's conjecture had been proved by L. de Branges. His proof is short and miraculous. It combines the theories of Loewner and Milin with a new ingredient from a totally unexpected source: a theorem of Askey and Gasper (Amer. J. Math. **98** (1976), 709–737, Theorem 3) which asserts that

$$\sum_{j=0}^{k} P_j^{(\alpha,0)}(x) > 0$$

for $-1 < x \leqq 1$ and $\alpha > -2$, where $P_j^{(\alpha,\beta)}$ denote the Jacobi polynomials.

Thus, some of the discussion of Bieberbach's conjecture below is obsolete, except insofar as it can serve to showcase the remarkableness of de Branges' achievement. Although its most famous problem has now been solved, the subject of univalent functions remains interesting, both for its own sake and for its connections with other branches of analysis, and Duren's book is an outstanding contribution to it.

In the language of classical complex function theory, "univalent" means one-to-one. Thus, the univalent functions of Duren's book are analytic functions which are one-one in some connected open subset of the complex plane, most often the unit disk $\mathbf{D} = \{z \in \mathbf{C}\colon |z| < 1\}$. Such functions effect a conformal mapping onto another domain $\Omega \subset \mathbf{C}$.

Much research in the subject, and most of this book, is devoted to the class S of univalent analytic functions f in $\mathbf{D}$ which satisfy the normalizations

$f(0)=0$, $f'(0)=1$. Thus, the Taylor series expansion of $f \in S$ around $z=0$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbf{D},$$

and $f(\mathbf{D})$ is a simply connected proper subdomain of $\mathbf{C}$ which contains $w=0$. S stands for Schlicht ("simple"), which is the German word for univalent.

The central theme in the study of univalent functions is the relation between the geometry of the image domain Ω and the analytic properties of the function f. Especially, what are the solutions of extremal problems? One presumes that the solutions to analytic extremal problems should correspond to extremal image domains. So, what are the extremal simply connected domains in $\mathbf{C}$? One candidate that comes to mind is the disk $\mathbf{D}$ itself. The corresponding function is the identity map $f(z)=z$. The opposite extreme from $\mathbf{D}$ seems reasonably to be the domain obtained by deleting from $\mathbf{C}$ a single radial slit $te^{i\alpha}$, $t_0 < t < \infty$. The normalizations in S imply that $t_0 = 1/4$. If we take $e^{i\alpha} = -1$, the omitted slit becomes part of the negative real axis and the corresponding function is

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n.$$

Here k stands for Koebe, who in 1907 was one of the first to study carefully the sort of extremal problems with which we are concerned in this essay. Note that the conformal mapping of $\mathbf{D}$ onto $\mathbf{C} \setminus \{te^{i\alpha}: \frac{1}{4} \leqslant t < \infty\}$ which belongs to S is $e^{i(\alpha+\pi)}k(ze^{-i(\alpha+\pi)})$, a "rotation" of k.

It is indeed the case that k and the identity are extremal for many problems. For example, fix $r \in (0,1)$ and consider the maximum modulus

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

The function in S with smallest $M(r,f)$ is the identity (apply the maximum principle to $f(z)/z$), whereas the ones with largest $M(r,f)$ are k and its rotations. This second result is part of the "distortion theorem", proved in this sharp form by Bieberbach in 1916, following earlier results by Koebe in 1909.

Another part of the distortion theorem states that on $|z|=r$ the function with smallest minimum modulus is k (along with its rotations), while the one with largest minimum modulus is again easily seen to be the identity. The distortion theorem was generalized by the reviewer in 1974 as follows:

$$(1) \qquad \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \leqq \int_{-\pi}^{\pi} |k(re^{i\theta})|^p \, d\theta,$$

for $f \in S$, $0 < r < 1$, and $-\infty < p < \infty$. That is, the Koebe function has largest integral means of all positive and negative orders. The main tool used in proving this theorem is the fact that a certain auxiliary function $(\log|f|)^*$ is subharmonic, a result which first came up in the solution of a growth problem in Nevanlinna's theory of meromorphic functions and has other applications elsewhere in function theory. A short survey of the $*$-function is in **[5]**.

Which function in S do you suppose has the coefficients a_n of largest modulus? In view of the results quoted above, there are good grounds for guessing it is k. This was done by Bieberbach in 1916.

BIEBERBACH'S CONJECTURE. $|a_n| \leqq n$, $n = 2, 3, \ldots, f \in S$.

This has been the principle stimulus for research on univalent functions in simply connected domains. As of this writing, it is still open.

To a large extent the story of univalent functions is the story of methods for attacking extremal problems. Especially, what can a method tell us about B.C. We shall give now a brief discussion of some of the significant partial results on B.C., along with a small inkling of the methods used to prove them, methods which have originated from a variety of sources inside and outside of classical function theory. More information can be found in the survey articles [**6** and **10**].

First of all, Bieberbach himself proved $|a_2| \leqq 2$ in 1916. Though nontrivial, this is a fairly elementary result which can be proved many ways using modern machinery.

The result $|a_3| \leqq 3$, due to Loewner (1923), is already very deep. Even now there is no easy proof. Loewner's proof is based on a parametric method which has many other applications. Here is one version, developed in [**20**]. Suppose that the complement of $f(\mathbf{D})$ consists of a single arc Γ starting at some $w_0 \in \mathbf{C}$ and ending at ∞. Such "slit mappings" are dense in S. Let $\gamma(t)$, $0 < t < \infty$, be a parameterization of Γ, and let $f(z, t)$ be the conformal map of $\mathbf{D}$ onto the complement of the shortened slit $\{\gamma(s)\colon\ t \leqq s < \infty\}$ with $f(0, t) = 0$, $(\partial f/\partial z)(0, t) > 0$. By Schwarz's lemma $(\partial f/\partial z)(0, t)$ increases with t. Reparameterizing Γ, if necessary, we may assume $f(z, t) = e^t z + a_2(t)z^2 + \cdots$.

Thus $f(z, 0) = f(z)$, and it turns out that $f(z, t)$ satisfies a partial differential equation

$$\frac{\partial f}{\partial t} = z\frac{\partial f}{\partial z}P,$$

where $P(z, t) = \sum_{n=0}^{\infty} c_n(t)z^n$ is an analytic function with positive real part. Now functions with positive real part are fairly easy to deal with, and one obtains $|a_3| \leqq 3$ by clever manipulations starting from the formula

$$a_n'(t) = \sum_{m=1}^{n-1} m a_m(t) c_{n-m}(t) + n a_n(t).$$

Loewner's theory provides one link between univalent functions and differential equations. Another was found in 1949 by Nehari, who made use of the notion of disconjugacy to give a criterion for univalence—if f is analytic in $\mathbf{D}$ and its Schwarzian derivative

$$\{f, z\} = (f''(z)/f'(z))' - \tfrac{1}{2}(f''(z)/f'(z))^2$$

satisfies $|\{f, z\}| \leqq 2(1 - |z|^2)^{-2}$, then f is univalent in $\mathbf{D}$. Ahlfors and Weill (1962) showed that if 2 is replaced by any smaller constant, then f can be extended to a quasiconformal homeomorphism of the whole plane onto itself. Quasiconformal mapping and Techmüller theory, which derives from the study of moduli of Riemann surfaces and uses q.c. maps as basic tools, have been two of the most active areas of classical function theory during the past thirty years. Accounts may be found in the survey articles [**3**, **7**, **8**, **9**, **11**, **14**], the books [**1**, **18**, and **19**], Gehring's review of Krushkal's book [**13**], and Ahlfors' collected papers [**4**]. We do not wish to discuss these subjects in detail, but will digress long enough to describe one interesting open problem about univalent functions.

Let B denote the Banach space of functions g analytic in $\mathbf{D}$ for which the norm

$$\|g\| = \sup_{\mathbf{D}} \left(1 - |z|^2\right)^2 |g(z)|$$

is finite. If $f \in S$ then its Schwarzian derivative belongs to B and satisfies $\|\{f, z\}\| \leqq 6$. This distortion theorem is due to Kraus (1932). The mapping $f \to \{f, z\}$ is injective modulo linear fractional transformations: $\{f, z\} \equiv \{g, z\}$ if and only if $g = (af + b)/(cf + d)$ for some $a, b, c, d \in \mathbf{C}$ with $ad - bc = 1$. The norm topology in B induces a topology in S called the Bers topology. Now consider the subset T of S whose members have q.c. extensions to the whole plane. T, or more often its image in B, is the "universal Teichmüller space". The Ahlfors-Weill theorem states that the image of T contains a ball. A later theorem of Ahlfors (1963) asserts that T is in fact open in the Bers topology, and then Gehring (1977) showed that, conversely, the interior of S is exactly T. Meanwhile, Bers had conjectured that T is dense in S, but Gehring disproved this in 1978 [**12**]. Now the question: Just what is the Bers closure of T in S? Ahlfors (1963) gave a very satisfying geometric characterization of "quasicircles", that is, of $\partial f(\mathbf{D})$ for $f \in T$ [**1**, p. 81], and it would be very nice to have a geometric description of the domains you can get to by taking limits of quasidisks.

Nehari's theorem has other offshoots as well. Duren-Romberg-Shields (1966) observed that if f is analytic in $\mathbf{D}$ with $|f''(z)/f'(z)| \leqq C(1 - |z|^2)^{-1}$ for sufficiently small C, then f satisfies Nehari's condition and, hence, is univalent. Becker (1972) used Loewner's theory to show $C = 1$ will do. It is not known what the largest possible C is. Analytic functions with $f'(z) \neq 0$ are "local quasi-isometries". Recently Gehring [**15**], motivated by work of F. John (1969), has characterized the "rigid" domains Ω in $\mathbf{C}$, that is, the ones with the property that every local quasi-isometry in Ω, not necessarily analytic, which is sufficiently close to the identity, must be globally univalent. The rigid domains turn out to be the quasidisks, thus adding another characterization of quasidisks to the seventeen discussed in [**14**].

We now resume our discussion of the Bieberbach conjecture. In 1955 Garabedian and Schiffer proved that $|a_4| \leqq 4$. They combined Loewner's method with a variational method invented by Schiffer in 1938, and required many pages of complicated calculations. Later, several simpler proofs were found using the "Grunsky inequalities" (1939), which are inequalities for quadratic forms derived from power series expansions of the two-variable analytic function $\log[(g(z_1) - g(z_2))/(z_1 - z_2)]$. Here $g(z) = 1/f(1/z)$, and $|z_1|, |z_2| > 1$. The proof of $|a_4| \leqq 4$ in Duren's book, due to G. V. Kuzmina, takes just two pages.

In 1968 Pederson and Ozawa independently proved $|a_6| \leqq 6$ by means of the Grunsky inequalities. As one might imagine, the details are formidable. Then in 1972 Pederson and Schiffer used the Garabedian-Schiffer inequalities, a generalization of the Grunsky inequalities, to prove $|a_5| \leqq 5$.

For $n \geqq 7$ it is not known whether $|a_n| \leqq n$.

It is known that $a_n = O(n)$ is the right order of magnitude. Littlewood proved in 1925 that $|a_n| \leqq en$. He deduced this via Cauchy's formula for a_n from his integral inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta \leqq \frac{r}{1-r}, \qquad 0 < r < 1,\ f \in S.$$

The sharp integral inequality (1) gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |k(re^{i\theta})|\, d\theta = \frac{r}{1-r^2},$$

from which Cauchy's formula leads to $|a_n| \leqq \frac{1}{2}en \leqq 1.37n$. This is the best one can do using considerations based solely on the "size" of f. Improvements require discovery of subtle cancellation properties. Such a method had been found by Milin in 1965, who combined the Grunsky inequalities with inequalities for coefficients of exponentiated power series. He was able to prove $|a_n| \leqq 1.243n$. FitzGerald (1972) used Schur's theorem about combinations of quadratic forms to find another way of exponentiating the Grunsky inequalities. The result is a set of quadratic inequalities amongst the $|a_n|^2$, a special case of which is a weighted sub-mean-value type inequality

$$|a_n|^4 \leqq \sum_{k=1}^{n} k|a_k|^2 + \sum_{k=n+1}^{2n-1} (2n-k)|a_k|^2.$$

This, together with the formula

$$\sum_{k=1}^{n} k^3 + \sum_{k=n+1}^{2n-1} (2n-k)k^2 = \left(\frac{7}{6}\right)^2 n^4 - \frac{1}{6}n^2 < \left(\frac{7}{6}\right)^2 n^4,$$

leads to the estimate $|a_n| \leqq (7/6)^{1/2} < 1.081n$.

D. Horowitz (1978) refined Fitzgerald's method to obtain

$$|a_n| < \left(\frac{1{,}659{,}164{,}137}{681{,}080{,}400}\right)^{1/14} < 1.0657n,$$

the best general bound now known.

The Bieberbach conjecture is true asymptotically, in a certain sense. Hayman, inspired by the Hardy-Littlewood circle method from number theory, proved in 1955 that $|a_n| \leq n$ for $n \geq n_0(f)$. Let $A_n = \sup_{f\in S}|a_n|$. Hayman showed in 1958 that $K_0 = \lim_{n\to\infty}(A_n/n)$ exists. His "asymptotic Bieberbach conjecture", that $K_0 = 1$, is still open. Recent work of D. Hamilton (1982), combined with a result of Nehari (1957), shows that $K_0 = 1$ is equivalent to a conjecture of Littlewood about coefficients of nonvanishing univalent functions in **D**.

The Bieberbach conjecture is true on the average, at least in certain senses. The L^2 case of the integral means inequality (1) shows that

$$\sum_1^\infty |a_n|^2 r^n \leqq \sum_1^\infty n^2 r^n, \qquad 0 < r < 1.$$

It is not known if B. C. is true in other average senses. For example, is it true that

$$\sum_{m=1}^{n} |a_m|^2\left(1 - \frac{m}{n}\right) \leqq \sum_{m=1}^{n} m^2\left(1 - \frac{m}{n}\right)?$$

B. C. is true in a certain sense for functions sufficiently near the Koebe function. Garabedian-Schiffer, for even n, and Bombieri, for odd n, proved in 1967 that there exists $\varepsilon_n > 0$ such that if $0 < |a_2 - 2| < \varepsilon_n$, then $\operatorname{Re} a_n < n$. The stronger local result, $0 < |a_n - 2| < \varepsilon_n$ implies $|a_n| < n$, has not been proved. On the other hand, B. C. is true for functions sufficiently far from k. If $|a_2| \leqq 1.64$ then $|a_n| \leqq n$ for all n. This result is due to Gong Sheng (1979), following earlier results of this type by Aharonov and Bshouty.

Finally, we mention that B. C. is known to be true for functions in various subclasses of S. The two easiest to describe are the "typically real" functions, those for which $f(\mathbf{D})$ is symmetric with respect to the real axis (Dieudonné, 1931), and the "starlike functions", those for which $f(\mathbf{D})$ intersects each ray from the origin in exactly one interval (R. Nevanlinna, 1921).

Duren's book contains a very beautiful and self-contained account of most of these topics, as well as much more. The only prerequisites are the basic graduate courses in real and complex analysis. Complete chapters are devoted to Loewner's parametric theory and Schiffer's variational theory. One chapter presents highlights from the Schaeffer-Spencer theory of the coefficient region, while part of another expounds the work of Ruscheweyh and Sheil-Small on convolutions and solution of the Pólya-Schoenberg conjecture (1974). The theories of Grunsky, Milin, and FitzGerald are all here, as is a proof of the reviewer's integral means theorem (1), including a proof of subharmonicity of the $*$-function.

Duren also discusses connections between univalent functions and concepts from linear space theory. Brickman (1970) proved that if f is an extreme point of the convex hull of S then the complement of $f(\mathbf{D})$ is a monotone arc, that is, intersects each circle $|w| = r$ at most once. For "support points", those functions which maximize some linear functional $f \to \operatorname{Re} L(f)$ over S, Pfluger (1971) used Schiffer's variational method to show that the complement of $f(\mathbf{D})$ is a monotone arc with strong additional properties. The arc must be analytic, and enjoys the $\pi/4$-property—at every point the angle between the tangent vector and the radius vector from the origin is at most $\pi/4$.

Hamilton **[17]** has developed a promising new techinque involving quasiconformal variations. One of his results is the existence of extreme points whose omitted arcs are not analytic. Thus, not every extreme point is a support point. It remains unknown whether every support point is an extreme point.

There exist some other very good recent books about univalent functions. Pommerenke's book [20], like Duren's, is a comprehensive text starting from scratch and is written at roughly the same level. There is considerable overlap between the two but also some significant differences. For example, Pommerenke's book does not contain proofs of (1) or the Ruscheweyh–Sheil-Small theorem, but it contains more information about boundary behavor of f and f'. Duren's book does not treat extremal length or Jenkins' general coefficient theorem, whereas Pommerenke's does. Pommerenke develops some of the connections with quasiconformal mapping, whereas Duren mentions them only in passing.

Schober's lecture notes [21] provide additional information about the linear space and q.c. connections. Ahlfors [2] gives a nice introduction to extremal length. A. W. Goodman [16] has recently written a book which devotes more attention to the elementary aspects of the subject, such as special classes, than do Duren and Pommerenke.

Duren's book is a joy to read, both physically, because of the neat bright type face, and intellectually, because of the inherent beauty of the subject and the authoritative yet sprightly way it is presented. As in his previous book, *Theory of H^p spaces* (1970), the writing style is decidedly user friendly. Great care has been taken with the prose, the facts, and the proofreading. The exercises range from simple to challenging. The index and bibliography are extensive. This is an excellent text for a graduate course and a worthy investment for both beginners and experts. We said before that the story of univalent functions is the story of methods. This book, which assembles so many known ones in attractive and suggestive fashion, should serve as inspiration for the future discovery of new ones.

REFERENCES

1. L. V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton, N. J., 1966.
2. ______, *Conformal invariants*, McGraw-Hill, New York, 1973.
3. ______, *Quasiconformal mappings, Teichmüller spaces, and Kleinian groups*, Proc. Internat. Congress Math. (Helsinki, 1978), Vol. 1, Academia Scientiarum Fennica, 1980, pp. 71–84.
4. ______, *Collected papers*, Vols. 1, 2, Birkhaüser, Boston, 1982.
5. A. Baernstein, *How the *-function solves extremal problems*, Proc. Internat. Congress Math. (Helsinki, 1978), Vol. 2, pp. 639–644.
6. ______, *Bieberbach's conjecture for tourists*, Harmonic Analysis (Proc Conf., Minneapolis, 1981), Lecture Notes in Math., vol. 908, Springer-Verlag, Berlin, 1982.
7. ______, *Topics in quasiconformal mappings*, Topics in Modern Harmonic Analysis (Proc. Conf. Milano, 1982), L. de Michele and F. Ricci, eds., Istituto Nazionale di Alta Matematica Francesco Sevani, Roma, 1983.
8. L. Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
9. ______, *Finite dimensional Teichmüller spaces*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), 131–172.
10. P. L. Duren, *Coefficients of univalent functions*, Bull. Amer. Math. Soc. **83** (1977), 891–909.
11. F. W. Gehring, *Some metric properties of quasi-conformal mappings*, Proc. Internat. Congress Math. (Vancouver, 1974), Vol. 2, Canadian Math. Congress, 1975, pp. 203–207.
12. ______, *Spirals and the universal Teichmüller space*, Acta Math. **141** (1978), 99–113.
13. ______, Review of [18], Bull Amer. Math. Soc. (N.S.) **3** (1980), 1062–1064.
14. ______, *Characteristic properties of quasidisks*, Les Presses Univ. Montréal, Montreal, 1982.
15. ______, *Injectivity of local quasi-isometries*, Comment. Math. Helv. **57** (1982), 202–220.

16. A. W. Goodman, *Univalent functions*, Vols. I, II, Mariner, Tampa, 1983.

17. D. H. Hamilton, *Extremal boundary problems for Schlicht functions*, preprint, Univ. of Maryland, 1984.

18. S. L. Krushkal, *Quasiconformal mappings and Riemann surfaces*, V. H. Winston, Washington, D. C., 1979.

19. O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, 2nd ed. Springer-Verlag, Berlin, 1973.

20. Ch. Pommerenke, *Univalent functions*, Vandenheock and Ruprecht, Göttingen, 1975.

21. G. Schober, *Univalent functions—selected topics*, Lecture Notes in Math., vol. 478, Springer-Verlag, Berlin, 1975.

ALBERT BAERNSTEIN II

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Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators, by Shmuel Agmon, Mathematical Notes, Vol. 29, Princeton University Press, Princeton, New Jersey, 1982, 118 pp., $10.50. ISBN 0-6910-8318-5

Square integrable eigenfunctions of the Schrödinger equation decay exponentially. More precisely, let

$$\tilde{H} = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_i + \sum_{i<j} V_{ij}(x_i - x_j), \qquad x_i \in \mathbf{R}^\nu,$$

be the Schrödinger Hamiltonian for N particles interacting with real pairwise potentials $V_{ij}(x_i - x_j)$, where $V_{ij}(x_i - x_j) \to 0$ (in some sense) as $x_i - x_j \to \infty$ in $\mathbf{R}^\nu$. Separating out the center of mass ($\tilde{H}$ itself has only continuous spectrum) one obtains the operator

$$H = -\Delta + \sum_{i<j} V_{ij}(x_i - x_j),$$

where Δ denotes the Laplacian on $L^2(X)$, $X = \{x = (x_1, \ldots, x_N)\colon \sum_{i=1}^N m_i x_i = 0\}$. If ϕ is an L^2 solution of $H\phi = E\phi$, and if E lies below the essential spectrum of H, then ϕ decays exponentially in the sense that there exist positive constants A and B for which $|\phi(x)| \leqslant Ae^{-B|x|}$. The phenomenon of exponential decay has long been recognized and was apparent already in Schrödinger's solution of the hydrogen atom, but is is only recently that a satisfactory mathematical theory for the problem has been developed.

There is a considerable chemical, physical, and mathematical literature on the subject, and we refer the reader to **[9, 7]**, and also the notes to Chapter XIII of **[14]**, for extensive historical and bibliographic information. Four general techniques have emerged.

(1) *Comparison methods* (see for example **[4, 5** and **3]**). These methods are based on the maximum principle for second order elliptic operators and are modelled, to a greater or lesser extent, on the standard proofs of such classical theorems of complex analysis as the Hadamard three-line theorem, the Phragmen–Lindelöf theorem, and so on.