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Mathematical scattering theory, by H. Baumgärtel and M. Wollenberg, Akademie-Verlag, Berlin, 1983, 449 pp., 98 DM.

Apparently it was Leonardo da Vinci who first mentioned the diffraction of light. By this he meant the illumination observed within the geometrical shadow of an opaque body. Huygens, then Young and Fresnel in the early nineteenth century, proposed that this was due to the interference effects of different parts of the light rays. Subsequently, other diffraction—or scattering—phenomena were discovered in acoustics, elasticity, quantum mechanics, and so on. A great impetus came from Maxwell's theory of electromagnetism. Mathematical formulations were given in classical times by Helmholtz, Kirchhoff, Rayleigh, and Sommerfeld.

What do these disparate physical situations have in common? First, all of them are described by a kind of wave equation. Second, you have an incident wave which comes in and gets disturbed or diffracted or scattered. What comes out is the reflected or refracted wave. The scattering process takes you from the incident wave u_{in} to the reflected wave u_{out}. It could happen that part of the incident wave is "trapped" or "bound" instead of scattered. The picture to keep in mind is that of a billiard ball bouncing off an irregularly shaped obstacle. It could bounce off cleanly or it could get trapped within an indentation of the obstacle. In order to describe the mathematical context, we have to become more specific.

EXAMPLE 1. In electromagnetism the basic equations are Maxwell's. Raleigh demonstrated that the reason the sky is blue is that light scatters off the water droplets in the atmosphere. But let's simplify and just take the ordinary wave equation. Say the incident wave is $u_{\text{in}}(x, t) = \exp[i\lambda(t - \omega \cdot x)]$ and the scatterer is an opaque body $B \subset \mathbf{R}^3$. Then we must solve

$$u_{tt} - \Delta u = 0 \quad \text{outside } B \tag{1}$$

with an appropriate boundary condition and an initial condition given by u_{in} for $t \ll 0$. We need to decompose the solution of (1) into harmonic plane waves (i.e., do a Fourier analysis), as well as take a limit as $t \to +\infty$. The scattering matrix tells you how much of the incident wave of frequency λ and direction ω goes into the reflected wave of frequency λ' and direction ω'. Part of the wave may be trapped. The wave-ray duality which goes back 150 years plays a prominent role. It is enjoying a revival with the advent of "microlocal analysis".

EXAMPLE 2. In two-body quantum theory one of the bodies is placed at the origin, so that only the state of the other body matters. This state is described by the Schrödinger equation

$$-i\frac{\partial u}{\partial t} = -\Delta u(x, t) + V(x)u(x, t), \qquad x \in \mathbf{R}^3, \tag{2}$$

where the potential function $V(x)$ plays the role of the scattering mechanism. For the hydrogen atom, $V(x) = c|x|^{-1}$. Basic to both the physics and the mathematics is the Hilbert space $\mathscr{H} = L^2(\mathbf{R}^3)$ and the unbounded operator $H = -\Delta + V$. If you know all about H, you know all about equation (2). The modern theory of quantum scattering goes back to Friedrichs' work on spectral theory in the 1930s and 1940s. It really got going with Kato's brilliant proof of the selfadjointness of H in 1951. The bound states are the solutions of (2) of the form $u = \phi(x)\exp(i\lambda t)$, with $\phi \in \mathscr{H}$. Thus, λ is an eigenvalue of H. Since there is no way that the bound states could "scatter", one works on the orthogonal complement $\mathscr{K}$ of the (often finite-dimensional) subspace of bound states. We use the notation $u(x,t) = [\exp(itH)f](x)$ for the solution of (2) which satisfies the initial condition $u(x,0) = f(x)$. In the same way we introduce the free Hamiltonian $H_0 = -\Delta$ and its evolution operator $\exp(itH_0)$. These are groups of unitary operators on $\mathscr{H}$. The fundamental mathematical problem is to ask for which vectors f, f_{in}, f_{out} in $\mathscr{H}$ do we have

$$(3) \qquad \left\| e^{itH}f - e^{itH_0}f_{\text{out/in}} \right\| \to 0 \quad \text{as } t \to \pm\infty,$$

where f_{out} goes with $+\infty$ (the future) and f_{in} goes with $-\infty$ (the past). If (3) holds for some triple we write $f_{\text{out}} = S\, f_{\text{in}}$, where S is the *scattering operator*. One wants to show that the limits (3) exist and then study the properties of S. (For the hydrogen atom the limits do not exist but require some modification.)

There is an intimate relation between scattering theory and spectral theory. One defines the wave operators $W_+(f_{\text{out}}) = f$ and $W_-(f_{\text{in}}) = f$. If in (3) we replace t by $t + s$, we see that both W_+ and W_- are intertwining operators for $\exp(isH)$ and $\exp(isH_0)$. Thus, $W_\pm H = H_0 W_\pm$. Actually this could be valid only on the subspace $\mathscr{K}$. In many cases one can prove that W_+ and W_- are unitary from $\mathscr{H}$ onto $\mathscr{K}$. Therefore, H_0 and the restriction of H to $\mathscr{K}$ are unitarily equivalent operators. This is the best method for studying the continuous spectrum of operators like H.

EXAMPLE 3. In relativisitic quantum mechanics the equations have to be Lorentz-invariant. A simple model equation is the Klein–Gordon equation (for mesons)

$$(4) \qquad \partial^2 u/\partial t^2 - \Delta u + m^2 u + g u^3 = 0.$$

If $g > 0$ there are no bound states, and one can ask whether (3) is valid. Here $\exp(itH)$ represents the (nonlinear) dynamics of (4) and $\exp(itH_0)$ the dynamics of the "free" equation with $g = 0$. Other model equations are Maxwell–Dirac for photons and electrons and Yang–Mills in unified field theories. The key property of the latter is its gauge invariance. The extensions of quantum mechanics to particles which move at relativistic speeds has been fraught with difficulties ever since the first attempts half a century ago. Heisenberg and Moller proposed in the 1940s that the scattering operator (or "S-matrix") should play a central role in the theory. S is somehow more intrinsic than H because it is determined from the scattering cross sections of beams of particles and thus is constructible from experimental data, at least in principle. In fact, S continues to play a major role in modern axiomatizations of quantum field theory.

EXAMPLE 4. Solitons are special solutions of nonlinear equations which play a role similar to the bound states of quantum theory. One example is the Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = 0, \qquad x \in \mathbf{R}, \tag{5}$$

where the solitons are solutions of the form $u = \phi(x - ct)$ for each speed $c > 0$. Their amazing property is that two solitons (for different c's) pass through each other essentially unscathed, behavior which you would expect for a linear equation but not for a nonlinear one. Another amazing property is that the spectrum of the Schrödinger operator $H = -d^2/dx^2 + u(\cdot, t)/6$ does not depend on t. In fact, the (inverse) scattering theory for H is intimately tied to the solution of (5). These two amazing properties are related, and the theory has many mathematical offshoots.

Our four examples illustrate how diverse (scattered!) scattering theory is. Rather than reviewing all its recent developments, which would be a lengthy task, I'll just list a few areas of scattering theory which are currently very active.

(i) *Inverse problems*. There are many kinds, stimulated by oil prospecting and other practical problems. Scattering data is observed. What mechanism (body, potential, oil pool, etc.) produces it? This is the most important and most difficult set of scattering problems.

(ii) *Multichannel scattering*. This is the quantum mechanical n-body problem.

(iii) *Scattering in gauge theories*.

(iv) *Location of the scattering poles*. The scattering matrix (see Example 1) extends meromorphically to complex frequencies. Its poles provide precise information about the scattering process.

(v) *Algebraic soliton theory*. Only certain special equations share the amazing properties of Example 4. Is there a simple algebraic characterization of all such equations?

The book under review is concerned with the kind of scattering theory of Example 2. After the intensive development of the past thirty years, the theory has reached a certain maturity. For instance, it is now known that a good scattering theory exists, provided $V(x)$ falls off strictly faster than $|x|^{-1}$ as $|x| \to \infty$. Furthermore, one knows how to modify the theory in the long-range case. It is characteristic of a mature theory that several extensive expositions appear. In this case there are the books of Reed–Simon, Amrein–Jauch–Sinha, Berthier, and now the book under review. What distinguishes the last is its level of abstraction and generality. As the authors write, "the aim of this book is to give a systematic and self-contained presentation of the Mathematical Scattering Theory within the framework of operator theory in Hilbert space".

Thus, the book deals with a Hilbert space $\mathscr{H}$ (or two) and a pair of selfadjoint operators H and H_0 on it. After an introduction to spectral theory and direct integrals (Part I), the book lays the groundwork in Part II by studying the algebra of operators A such that the limits of $\exp(itH)A\exp(itH)$ exist as $t \to \pm\infty$. In Part III the (two-space) wave and scattering operators are introduced. Included here are the stationary theory, the invariance principle for W_+ and W_-, and introductions to multichannel scattering, abstract quantum fields, and the Lax–Phillips theory. One result of the authors is the inverse problem. Roughly, if operators H_0 and S are given and they commute, then there exists an operator H; however, H is highly nonunique in this general setup.

Finally, in Part IV, well past the midpoint of the book, come the first theorems which assert the existence of the limits (3). Of course, the groundwork has been carefully laid, so the proofs are now efficient. There are chapters on stationary methods, time-falloff methods, trace-class methods, and smooth perturbations. These methods are compared, and the main applications to different operators are given. In Part V the authors discuss formulas for the scattering amplitude $\hat{S}(\lambda)$. This means $\mathscr{H}$ is represented as on L^2 space (in the "frequency" variable λ) with vector values so that the free Hamiltonian H_0 is multiplication by λ. Then S is given by the operator-valued function $\hat{S}(\lambda)$, which has a meromorphic continuation into the complex plane. There is a brief discussion of its poles, the phase shift, and the scattering cross section.

The authors give us a very well-organized and complete exposition of the most abstract parts of scattering theory, to which they have been active contributors. There are historical notes and a very extensive bibliography. Two things need improvement: the index is so small as to be almost useless (although the table of contents is detailed), and the quality of the paper is poor. The book is meant to be read systematically. If you really want to understand a proof, you have to do a backwards search. Thus, the book is a very useful contribution to the literature, in that it brings together all aspects of the abstract theory, but most nonspecialists would find it too detailed. The authors wrote that "applications of the general results to special scattering systems... are treated mainly for purposes of illustration". They are presented very concisely and in small print. But the juiciest parts of the subject lie in the applications to differential operators, and so the book lacks a certain balance. The student who reads it might come away with the impression that scattering theory is a special branch of operator theory, that it is entirely concerned with the existence of the limits (3), and that its only serious application is to quantum scattering. Therefore this book should be supplemented by references which focus on other aspects of scattering theory and on its physical roots.

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Éléments de mathématique. Algèbre commutative, by N. Bourbaki, Chapitres 8 et 9, Masson, Paris, 1983, 200 pp., 150F. ISBN 2-2257-8716-6

Dimension theory of local rings, originated by Krull [**1**], can be regarded as the start of the theory of local rings. Since then many authors have contributed to the theory of local rings, including Chevalley [**2**, **3**], Cohen [**4**], Samuel [**5**], Serre [**6**] and Nagata [**7**, **8**]. Cohen proved the structure theorem of complete local rings, and Samuel constructed a good multiplicity theory of local rings.