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GEL'FAND'S PROBLEM ON UNITARY REPRESENTATIONS ASSOCIATED WITH DISCRETE SUBGROUPS OF $\mathrm{PSL}_2(\mathbf{R})$

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In 1978 M.-F. Vignéras [**10**] gave a negative answer to the question posed by I. M. Gel'fand [**2**], who asked if the induced representation of $\mathrm{PSL}_2(\mathbf{R})$ on $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbf{R}))$ determines a discrete subgroup Γ up to conjugation. She constructed explicitly two nonconjugate discrete groups arising from indefinite quaternion algebras defined over number fields giving rise to isomorphic induced representations. Such examples are necessarily quite sporadic since there are only finitely many conjugacy classes of arithmetic groups with a fixed signature; see K. Takeuchi [**9**]. The purpose of this note is to give, in a rather simple way, a large family of (nonarithmetic) discrete groups that are not determined by their induced representations. The key idea is to reduce the problem to the case of finite groups where the situations are simple and well understood. We should point out that a similar idea can be applied to constructions of various isospectral Riemannian manifolds [**8**].

Our construction is based on the following proposition, which follows from standard facts about induced representations.

PROPOSITION. *Let G be a locally compact topological group, and let $\Gamma \subset \Gamma_1,\ \Gamma_2 \subset \Gamma_0$ be discrete subgroups, with Γ normal and of finite index in Γ_0. Then if the subgroups $\mathcal{H}_i = \Gamma_i/\Gamma,\ i = 1,2,$ of $\mathcal{G} = \Gamma_0/\Gamma$ meet each conjugacy class of $\mathcal{G}$ in the same number of elements, the representations of G on the spaces $L^2(\Gamma\backslash G),\ i = 1,2,$ are unitarily equivalent.*

To construct nonconjugate Γ_1 and Γ_2 in $\mathrm{PSL}_2(\mathbf{R})$, we first choose an appropriate triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ with the same induced representation $\mathrm{Ind}_{\mathcal{H}_1}^{\mathcal{G}}(1) \equiv \mathrm{Ind}_{\mathcal{H}_2}^{\mathcal{G}}(1)$. For instance, we let $\mathcal{G}$ be the semidirect product $(\mathbf{Z}/8\mathbf{Z})^\times \cdot (\mathbf{Z}/8\mathbf{Z})$ and set $\mathcal{H}_1 = \{(1,0),(3,0),(5,0),(7,0)\}$, $\mathcal{H}_2 = \{(1,0),(3,4),(5,4),(7,0)\}$. It is easy to check that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not conjugate in $\mathcal{G}$, and each conjugacy class of $\mathcal{G}$ meets $\mathcal{H}_1$ and $\mathcal{H}_2$ in the same number of elements.

We then take a torsion-free discrete subgroup $\Gamma_0 \subset \mathrm{PSL}_2(\mathbf{R})$ satisfying the following conditions:

(i) the genus of the Riemann surface $\Gamma_0\backslash\mathrm{PSL}_2(\mathbf{R})/\mathrm{SO}(2)$ is greater than two;

(ii) Γ_0 is maximal in $\mathrm{PGL}_2(\mathbf{R})$; and

(iii) Γ_0 is nonarithmetic.

Since the fundamental group of a Riemann surface of genus k has a free group on k generators as quotient, we may always, if k is large enough, find a sur-

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jective homomorphism $\varphi\colon \Gamma_0 \to \mathcal{G}$. Since the example of $\mathcal{G}$ given in the previous paragraph has three generators, for it we may take $k = 3$. If we put $\Gamma_i = \varphi^{-1}(\mathcal{H}_i)$, then, in view of the Proposition, Γ_1 and Γ_2 give rise to isomorphic representations. But Γ_1 and Γ_2 are not conjugate in $\mathrm{PGL}_2(\mathbf{R})$. In fact, if $g\Gamma_1 g^{-1} = \Gamma_2$ for some $g \in \mathrm{PGL}_2(\mathbf{R})$, then $g\Gamma_0 g^{-1}$ is commensurable with Γ_0. A result announced by G. A. Margoulis [**6**] implies that the commensurability group $C(\Gamma_0) = \{h \in \mathrm{PGL}_2(\mathbf{R}); h\Gamma_0 h^{-1}$ is commensurable with $\Gamma_0\}$ is discrete provided that Γ_0 is nonarithmetic. Since $C(\Gamma_0) \supset \Gamma_0$, from the maximality of Γ_0, it follows that $g \in \Gamma_0$; hence, $\varphi(g)\mathcal{H}_1\varphi(g)^{-1} = \mathcal{H}_2$. This is a contradiction.

Existence of "many" Γ_0 satisfying (i)–(iii) is guaranteed by results of L. Greenberg [**4**], A. M. Macbeath and D. Singerman [**5**], and K. Takeuchi [**9**]. In fact, by [**4**], generic Γ_0 are maximal in $\mathrm{PSL}_2\mathbf{R}$). If such a Γ_0 is not maximal in $\mathrm{PGL}_2(\mathbf{R})$, then the normalizer $N(\Gamma_0)$ of Γ_0 in $\mathrm{PGL}_2(\mathbf{R})$ is strictly bigger than Γ_0, so that the isometry group $N(\Gamma_0)/\Gamma_0$ of the surface $\Gamma_0\backslash\mathrm{PSL}_2(\mathbf{R})/\mathrm{SO}(2)$ is not trivial. On the other hand, by [**5**], the isometry group is trivial for generic Γ_0. Combining these facts with finiteness of arithmetic groups [**9**], we get the genericity of Γ_0 satisfying (i)–(iii).

REMARK. (a) For the above groups Γ_i the genus of the surface $\Gamma_i\backslash\mathrm{PSL}_2(\mathbf{R})/\mathrm{SO}(2)$ equals $8k - 7$ ($k \geq 3$). The examples given by Vignéras have much bigger genus.

(b) Examples of $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ satisfying the condition in the Proposition have been used by number-theorists to construct nonisomorphic number fields with the same Dedekind zeta function (see, for instance, [**7**]). It is also known that many examples of the triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ arise from simple algebraic groups: If $\mathcal{G}$ is a reductive algebraic group and $\mathcal{H}_1, \mathcal{H}_2$ are nonconjugate but associate parabolic subgroups, then $\mathrm{Ind}_{\mathcal{H}_i}^{\mathcal{G}}(1)$, $i = 1, 2$, are isomorphic.

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