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UNITARY DUAL OF p-ADIC $GL(n)$. PROOF OF BERNSTEIN CONJECTURES

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1. Introduction. A fundamental problem of harmonic analysis on a locally compact group G is the description of the dual object $\hat{G}$ of G, i.e. description of the set of all equivalence classes of irreducible unitary representations of G. If G is a connected reductive p-adic group then $\hat{G}$ is in the natural bijection with the subset of all unitarizable classes in the set $\tilde{G}$ consisting of all equivalence classes of irreducible smooth representations. In this way the problem of parametrizing $\hat{G}$ breaks into two problems, the problem of describing the nonunitary dual $\tilde{G}$ and the problem of identifying unitarizable classes in $\tilde{G}$. The first problem has been studied much more than the second one, which is solved completely only for groups $SL(2)$, $GL(2)$. In the case of reductive Lie groups, the second problem has been solved only for some groups of low ranks. This paper announces the complete solution of the second problem for the case of the general linear group $GL(n)$ over a p-adic field F of characteristic zero (we describe Langlands parameters of $GL(n, F)\hat{}$). Proof of Bernstein conjectures on unitarizability in **[1]** is also announced.

2. Main results. Let R_n be the Grothendieck group of the category of all smooth representations of $GL(n, F)$ of finite length. We consider $GL(n, F)\tilde{} \subseteq R_n$. Set $R = \oplus R_n$ $(n \geqslant 0)$, $\mathrm{Irr} = \bigcup GL(n, F)\tilde{}$ $(n \geqslant 0)$ and $\mathrm{Irr}^u = \bigcup GL(n, F)\hat{}$ $(n \geqslant 0)$. The induction functor $(\tau, \sigma) \mapsto \tau \times \sigma$ defines a structure of the commutative graded ring on R (see **[8]**). Let D_0 be the subset of all square integrable representations in Irr. The character $g \mapsto |\det g|$ of $GL(n, F)$ is denoted by ν. Set $D = \{\nu^\alpha\delta;\ \alpha \in \mathbf{R}, \delta \in D_0\}$. The set of all finite multisets

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in D is denoted by $M(D)$ (see [**8**]). Let $a = (\delta_1, \ldots, \delta_n) \in M(D)$. Then $\delta_i = \nu^{\alpha_i}\delta_i^0$ for some $\alpha_i \in \mathbf{R}$, $\delta_i^0 \in D_0$. After renumbering, we may suppose that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. The representation $\delta_1 \times \delta_2 \times \cdots \times \delta_n$ has a unique irreducible quotient which is denoted by $L(a)$ (see [**3**]). Now $a \mapsto L(a)$, $M(D) \to \mathrm{Irr}$ is a bijection. Let σ be in Irr, and let σ^+ denote the Hermitian (complex conjugate) contragredient of σ. Set $\Pi(\sigma, \alpha) = \nu^\alpha\sigma \times \nu^{-\alpha}\sigma^+$ for $\alpha \in \mathbf{R}$. For $n \in \mathbf{Z}$, $n \geqslant 1$ and $\delta \in D_0$ set

$$u(\delta, n) = L(\nu^p\delta, \nu^{p-1}\delta, \ldots, \nu^{-p}\delta), \quad \text{where } p = (n-1)/2.$$

THEOREM 1. *Let* $B = \{u(\delta, n), \Pi(u(\delta, n), \alpha); \delta \in D_0, n \geqslant 1, 0 < \alpha < 1/2\}$.

(i) *If* $\sigma_1, \ldots, \sigma_r \in B$, *then* $\sigma_1 \times \cdots \times \sigma_r \in \mathrm{Irr}^u$.

(ii) *If* $\pi \in \mathrm{Irr}^u$, *then there exists* $\tau_1, \ldots, \tau_s \in B$, *unique up to a permutation, such that* $\pi = \tau_1 \times \cdots \times \tau_s$.

A representation $\pi \in \mathrm{Irr}$ is called rigid if there exist $\delta_1, \ldots, \delta_n \in D_0$, $\alpha_1, \ldots, \alpha_n \in (1/2)\mathbf{Z}$ so that $\pi = L(\nu^{\alpha_1}\delta_1, \ldots, \nu^{\alpha_n}\delta_n)$. The following theorem implies the validity of Bernstein conjecture 8.6 of [**1**].

THEOREM 2. (i) *Let* $\sigma \in \mathrm{Irr}$. *If* $\Pi(\sigma, \alpha)$ *is an irreducible unitarizable representation for all* $-1/2 < \alpha < 1/2$, *then* σ *is a unitarizable rigid representation.*

(ii) *Let* $\sigma \in \mathrm{Irr}$ *be a rigid representation. If* $\Pi(\sigma, \alpha)$ *is an irreducible unitarizable representation for some* $0 < \alpha < 1/2$, *then there exist* $\sigma_1, \sigma_2 \in \mathrm{Irr}^u$ *so that* $\sigma = \sigma_1 \times \nu^{-1/2}\sigma_2$.

A. V. Zelevinsky introduced an involutive automorphism ${}^t\colon R \to R$ (9.12 of [**8**]). The following theorem is just Conjecture 8.10 of [**1**].

THEOREM 3. *If* $\pi \in \mathrm{Irr}^u$ *then* $\pi^t \in \mathrm{Irr}^u$.

Using [**3**] one can, from Theorems 1 and 3, describe the unitary dual of $GL(n, F)$ in Zelevinsky classification.

3. Outline of proofs. From [**8**] one obtains that R is a $\mathbf{Z}$-polynomial algebra over indeterminates D. We can consider $u(\delta, n)$ as polynomials in R.

PROPOSITION 1. *The* $u(\delta, n)$, *as polynomials in elements of* D, *are irreducible.*

The proof of the proposition uses results on composition series in [**8**] and interpretation of them in [**3**]. The proof is based on the fact that the degrees of $u(\delta, n)$ in indeterminates $\nu^{(n-1)/2-i}\delta$ are 1.

THEOREM 4. *Representations* $u(\delta, n)$ *are unitarizable.*

SKETCH OF PROOF. Choose a number field k so that F is the completion of k at some place. By the proof of Proposition 5.15 of [**4**], δ is a factor of some irreducible cuspidal automorphic representation σ of $GL(m, A)$, where A is the Adele ring of k. Now §2 of [**2**] gives a construction of an element Π of residual spectrum of $GL(mn, A)$, from σ. Now $u(\delta, n)$ is a factor of Π, so it is unitarizable.

The following lemma is a consequence of [**1** and **3**].

LEMMA 1. *Let $a, b \in M(D)$. If $L(a)$ and $L(b)$ are unitarizable then $L(a) \times L(b) = L(a+b)$.*

SKETCH OF PROOF OF THEOREM 1. By [**1**] and Theorem 4 representations $\Pi(u(\delta, n), \alpha)$ are irreducible and unitarizable for $-1/2 < \alpha < 1/2$. Thus $B \subseteq \mathrm{Irr}^u$. By Lemma 1, Irr^u is a multiplicative semigroup. Let $X(B)$ be the subsemigroup of Irr^u generated by B. Proposition 1 implies that it is enough to prove that $\mathrm{Irr}^u \subseteq X(B)$. Let $\tau \in \mathrm{Irr}^u$. Choose $\delta_i \in D$ so that $\tau = L(\sigma_1, \dots, \sigma_m)$. Since τ is Hermitian one obtains that

$$(\sigma_1, \dots, \sigma_m) = (\nu^{\alpha_1}\delta_1, \nu^{-\alpha_1}\delta_1, \nu^{\alpha_2}\delta_2, \nu^{-\alpha_2}\delta_2, \dots, \nu^{\alpha_r}\delta_r, \nu^{-\alpha_r}\delta_r, \gamma_1, \dots, \gamma_v)$$

where $\delta_i, \gamma_j \in D_0$, $\alpha_i \in \mathbf{R}$, $\alpha_i > 0$. After a suitable renumbering we can choose w, $0 \leqslant w \leqslant r$, so that $\alpha_i \in (1/2)\mathbf{Z}$ for $1 \leqslant i \leqslant w$ and $\alpha_i \notin (1/2)\mathbf{Z}$ for $w < i \leqslant r$. Let $\alpha_i = \beta_i + m_i$, for $w < i \leqslant r$, so that $0 < \beta_i < 1/2$ and $m_i \in (1/2)\mathbf{Z}$. For $w < i \leqslant r$ we can find integers t_i and $\varepsilon(i)$, $\chi(i) \in \{-1/2, 0\}$ so that $\nu^{m_i}\delta_i + \operatorname{supp} \nu^{\varepsilon(i)} u(\delta_i, t_i) = \sup \nu^{\chi(i)} u(\delta_i, t_i + 1)$. Using Lemma 1, we compute

$$\begin{aligned}
&\tau \times u(\delta_1, 2\alpha_1 - 1) \times \cdots \times u(\delta_w, 2\alpha_w - 1) \\
&\quad \times \Pi\big(\nu^{\varepsilon(w+1)} u(\delta_{w+1}, t_{w+1}), \beta_{w+1}\big) \times \cdots \times \Pi\big(\nu^{\varepsilon(r)} u(\delta_r, t_r), \beta_r\big) \\
&= \gamma_1 \times \cdots \times \gamma_v \times u(\delta_1, 2\alpha_1 + 1) \times \cdots \times u(\delta_w, 2\alpha_w + 1) \\
&\quad \times \Pi\big(\nu^{\chi(w+1)} u(\delta_{w+1}, t_{w+1} + 1), \beta_{w+1}\big) \times \cdots \times \Pi\big(\nu^{\chi(r)} u(\delta_r, t_r + 1), \beta_r\big).
\end{aligned}$$

Since τ is Hermitian, Proposition 1 implies $\tau \in X(B)$.

The proof of Theorem 1 is similar to the above proof.

SKETCH OF PROOF OF THEOREM 3. We construct recursively a family X_n, $n \geqslant 0$, such that: $X_n \subseteq \mathrm{Irr}^u$, $(X_n)^t \subseteq \mathrm{Irr}^u$, X_n consists of rigid representations. Let $X_0 = D_0$. Let X_{n+1} be the set of all composition factors of representations $\Pi(\sigma, 1/2)$, $\sigma \in X_n$. Using [**1** and **5**] one obtains $X_n \subseteq \mathrm{Irr}^u$, $(X_n)^t \subseteq \mathrm{Irr}^u$. Let $X = \bigcup X_n (n \geqslant 0)$ and $Y = X \cup \{\Pi(\sigma, \alpha); \sigma \in X, 0 < \alpha < 1/2\}$. Now $Y \subseteq \mathrm{Irr}^u$ and $Y^t \subseteq \mathrm{Irr}^u$. Let $\pi \in \mathrm{Irr}^u$. A combinatorial argument, similar to the combinatorial argument used in the proof of Theorem 1, provides the existence of $\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_m \in Y$ such that $\pi \times \sigma_1 \times \cdots \times \sigma_n = \tau_1 \times \cdots \times \tau_m$. Thus $\pi^t \times \sigma_1^t \times \cdots \times \sigma_n^t$ is unitarizable. Now 8.2 of [**1**] implies $\pi^t \in \mathrm{Irr}^u$.

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