

DEHN SURGERY ON KNOTS

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Let M be a compact, connected, orientable, irreducible 3-manifold such that ∂M is a torus. An isotopy class c of unoriented simple closed curves in ∂M will be called a *slope*. A closed 3-manifold $M(c)$ may be constructed by attaching a solid torus J to M so that c bounds a disk in J .

If c and d are two slopes, we denote their (minimal) geometric intersection number by $\Delta(c, d)$.

THEOREM. *Suppose that M is not a Seifert fibered space. If $\pi_1(M(c))$ and $\pi_1(M(d))$ are cyclic, then $\Delta(c, d) \leq 1$. In particular, there are at most three slopes c such that $\pi_1(M(c))$ is cyclic.*

This result is sharp; Fintushel–Stern and Berge have given examples of hyperbolic knots in S^3 for which two Dehn surgeries give lens spaces.

In the statements of the following corollaries we use rational numbers as in [R] to parametrize the nontrivial Dehn surgeries on a knot K in S^3 . We will denote by $K(r)$ the result of r -surgery on K .

COROLLARY 1. *If K is not a torus knot and $r \in \mathbf{Q}$, then $\pi_1(K(r))$ can be cyclic only if r is an integer. Moreover, there are at most two such integers r , and if there are two then they must be successive.*

COROLLARY 2. *If K is a nontrivial knot and $r \in \mathbf{Q}$ is not equal to 1 or -1 then $K(r)$ is not simply-connected. Moreover, $K(1)$ and $K(-1)$ cannot both be simply-connected.*

COROLLARY 3. *Up to unoriented equivalence, there are at most two knots whose complements are of a given topological type.*

COROLLARY 4. *If K is a nontrivial amphicheiral knot and $r \in \mathbf{Q} - \{0\}$, then $\pi_1(K(r))$ is not cyclic. In particular, K has Property P.*

COROLLARY 5. *Knots of Arf invariant 1 are determined up to unoriented equivalence by their complements.*

Whitten [W], using work of Johannson [Jo1], shows that Corollary 1 implies the following result.

COROLLARY 6. *Prime knots with isomorphic groups have homeomorphic complements.*

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The theorem states that, with a very small set of exceptions, the group $\pi_1(M(c))$ is not cyclic. We prove this by showing that either

- (*) there exists an incompressible surface in $M(c)$; or
- (**) there exists a representation of $\pi_1(M(c))$ into $PSL_2(\mathbb{C})$ with non-cyclic image.

The proof reduces to the case where M is atoroidal, using part (a) of

PROPOSITION 1. (a) *If M contains an incompressible nonperipheral torus which compresses in $M(c)$ and $M(d)$, then either $\Delta(c, d) \leq 1$ or $(M, \partial M)$ is cabled in the sense of [GL].*

(b) *Suppose that $\dim H_1(M; \mathbb{Q}) > 1$. If $M(c)$ and $M(d)$ have cyclic first homology groups and are not Haken manifolds then $\Delta(c, d) \leq 1$.*

The proof of Proposition 1 is a combinatorial analysis of the intersection of the two planar surfaces in M corresponding, in (a), to the compressing disks for the torus in $M(c)$ and $M(d)$, and, in (b), to the nonseparating 2-spheres in $M(c)$ and $M(d)$ which would exist if $M(c)$ and $M(d)$ were not Haken.

We define a slope c to be a *boundary slope* if M contains an incompressible nonperipheral surface F with $\partial F \neq \emptyset$ such that each component of ∂F has slope c . The next proposition, together with Proposition 1 (b), establishes the inequality in the conclusion of the theorem when one of the slopes is a boundary slope.

PROPOSITION 2. *If c is a boundary slope and $\dim H_1(M; \mathbb{Q}) = 1$, then either*

- (i) *$M(c)$ is a Haken manifold; or*
- (ii) *$M(c)$ is a connected sum of two (nontrivial) lens spaces; or*
- (iii) *M contains a closed incompressible surface which remains incompressible in $M(d)$ whenever $\Delta(c, d) > 1$.*

For the proof of Proposition 2, let F be an incompressible nonperipheral surface in M with $\partial F \neq \emptyset$ having boundary slope c and with the minimal number of boundary components. Consider the manifold X obtained by cutting M along F . If there are enough compressing disks for ∂X in X , one shows either that the capped-off surface \hat{F} in $M(c)$ is an incompressible surface of positive genus and (i) holds, or that \hat{F} is an essential 2-sphere and (ii) holds. (The proof uses an extension of a result of Jaco [Ja] and Johannson [Jo2] giving conditions under which the addition of a 2-handle to a 3-manifold will yield a boundary-irreducible manifold.) Otherwise X , and hence M , contains a closed incompressible surface which is shown, by a refinement of the combinatorial analysis used in the proof of Proposition 1, to remain incompressible in $M(d)$ if $\Delta(c, d) > 1$. This gives conclusion (iii).

Finally, we consider the case that M is atoroidal and that c and d are nonboundary slopes. Thurston's Geometrization Theorem implies that the interior of M has a hyperbolic structure of finite volume. We define, as in [CS], a complex affine curve X in the space of characters of representations of $\pi_1(M)$ in $SL_2(\mathbb{C})$. We identify $L = H_1(\partial M; \mathbb{Z})$ with a lattice in the vector $V = H_1(\partial M; \mathbb{R})$. Let $e: L \rightarrow \pi_1(M)$ denote the inverse of the Hurewicz isomorphism followed by the inclusion $\pi_1(\partial M) \rightarrow \pi_1(M)$. Each $\gamma \in L$ defines a regular function $I_\gamma: X \rightarrow \mathbb{C}$ by $I_\gamma(\chi) = \chi(e(\gamma))$ (cf. [CS]). One shows that

there is a piecewise linear norm $\|\cdot\|$ on V such that for each γ in the lattice $L \subset V$, degree $I_\gamma = \|\gamma\|$. Then the ball of radius $m = \min_{0 \neq \gamma \in L} (\|\gamma\|)$ is a convex polygon B such that $B = -B$, and the interior of B contains no points of L . One concludes that, in terms of the natural area element on V , B has area at most 4.

We shall identify a slope with a pair $\{\pm\gamma\}$ of primitive elements of L . The following result is proved by the techniques of [CS].

PROPOSITION 3. (a) *Each vertex of B is a rational multiple of $\gamma \in L$, where $\{\pm\gamma\}$ is a boundary slope.*

(b) *If c is a nonboundary slope then either $c = \{\pm\gamma\}$, with $\gamma \in B$, or else one of the conclusions (*) or (**) holds for $M(c)$.*

Suppose now that $c = \{\pm\gamma\}$ and $d = \{\pm\delta\}$ are nonboundary slopes and that $M(c)$ and $M(d)$ satisfy neither (*) nor (**). Consider the parallelogram $P \subset V$ with vertices $\pm\gamma$ and $\pm\delta$. By (a) we have

$$\Delta(c, d) = \frac{1}{2} \text{Area } P \leq \frac{1}{2} \text{Area } B \leq 2$$

and equality would imply that γ and δ are vertices of B . By (b) we would have that c and d were boundary slopes, a contradiction. This completes the proof of the theorem.

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