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IRREDUCIBLE REPRESENTATIONS OF INFINITE-DIMENSIONAL TRANSFORMATION GROUPS AND LIE ALGEBRAS

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1. Introduction. We say that a Lie algebra $\mathfrak{L}$ of vector fields on a smooth manifold M is **transitive** provided that for each point $p \in M$, the vectors $\{X(p): X \in \mathfrak{L}\}$ form the tangent space at p. The algebra $\mathfrak{L}$ is **doubly transitive** if its natural lifting $\mathfrak{L} \oplus \mathfrak{L} = \{X \oplus X: X \in \mathfrak{L}\}$ of $\mathfrak{L}$ to $M \times M$ is transitive on the complement of the diagonal Δ. Higher orders of transitivity are defined analogously. (Just as the full group of diffeomorphisms of a manifold M is n-fold transitive for all n, so is its Lie algebra of vector fields; but the fact about the algebra is far easier to establish.) We are able to exploit the high degree of transitivity of many natural Lie algebras of vector fields to establish irreducibility and inequivalence of certain of their "geometric" or "induced" representations, regarded as unitary representations of the corresponding infinite-dimensional Lie transformation groups. Our technique is a direct descendant of a classical theorem of Burnside on permutation groups.

The applications include much simpler proofs of some of the results of the Soviet school on unitary representations of the group of diffeomorphisms [**10**]. We also get significant generalizations of the pioneering results of Léon van Hove [**9**] on what is now known as "prequantization", i.e., representations of the Poisson bracket Lie algebra of a symplectic manifold. The algebraic technique seems quite fruitful—this is not exactly a surprise—and other applications are forthcoming.

Some of our results were mentioned briefly in [**3**]. Full details will appear elsewhere.

2. The main theorems. If G is a group acting on a discrete set M, then the corresponding representation of G on $l^2(M)$ is irreducible on the orthogonal complement of the constants if and only if the group action on M is doubly transitive. This is essentially due to Burnside (cf. [**2**, p. 249]), and may also be extracted from the work of Mackey [**4**]. The proof is quite simple: If T is an intertwining operator with kernel $k(x, y)$, then we must have the invariance property $k(x, y) \equiv k(x \cdot g, y \cdot g)$ for $x, y \in M$ and $g \in G$. By double transitivity, $M \times M$ decomposes into two G-orbits (the diagonal Δ and its complement) on which the kernel k is constant. Hence k is a linear combination of the

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identity I and the orthogonal projection P onto the constants. Of course $P = 0$ if M is infinite.

Our main results are analogues of Burnside's theorem. The proofs are in the same spirit, formally. Of course, the analytic details are more involved. In particular, we make use of the Schwartz kernel theorem to study the intertwining operators.

THEOREM 1. *Let M be a connected finite-dimensional manifold with a smooth measure μ. Let $\mathfrak{L}$ be a doubly transitive Lie algebra of divergence-free vector fields on M. Suppose that each $X \in \mathfrak{L}$ defines an essentially skew-adjoint operator, with domain $C_0^\infty(M)$, in $L^2(M, \mu)$. (This roughly means that the vector field X has a complete flow.)*

Then the corresponding family of unitary operators $\{\exp(X)\colon X \in \mathfrak{L}\}$ acts irreducibly on the orthogonal complement of the constants; this is all of L^2 if $\mu(M)$ is infinite.

Now let $\mathfrak{L}$ be an abstract Lie algebra and let $A\colon \mathfrak{L} \to \mathrm{Vect}(M)$ be a Lie homomorphism of $\mathfrak{L}$ into the Lie algebra of vector fields on M. Let $\rho\colon \mathfrak{L} \to C^\infty(M)$ be a linear mapping, and suppose that ρ satisfies the "cocycle" identity

$$\rho([X, Y]) = A(X) \cdot \rho(Y) - A(Y) \cdot \rho(X). \tag{1}$$

Then $B(X) = A(X) + \rho(X)$ is also a homomorphism from $\mathfrak{L}$ into the differential operators on M. We say that the **multiplier** is **trivial** if ρ corresponds to a 1-form on M (in which case the identity (1) essentially says that $d\rho = 0$, so that the representation B is locally equivalent to A).

THEOREM 2. *Let $X \to A(X)$ be a doubly-transitive skew-adjoint representation of $\mathfrak{L}$ on M. Assume that ρ is a nontrivial multiplier for A. Set $B = A + \rho$, as above. Then (in the sense of Theorem 1) the representation B of $\mathfrak{L}$ on $L^2(M, \mu)$ is irreducible.*

3. Applications. A. Let M be a connected manifold, μ a smooth measure, and suppose for simplicity that $\mu(M) = \infty$. Let $\mathfrak{L} \subset \mathrm{Vect}(M)$ be the Lie algebra of compactly supported divergenceless vector fields; formally $\mathfrak{L}$ is the Lie algebra of the group $G = \mathrm{Diff}_0(M, \mu)$ of volume-preserving diffeomorphisms g such that g = identity outside some compact set (depending on g). It is easy to see that, if the dimension of M is greater than one, then $\mathfrak{L}$ is n-fold transitive for all n. Hence the natural, "Koopman", representation U of G on L^2 is irreducible by Theorem 1. The techniques used in the proof of Theorem 1 may also be applied to study the higher tensor powers of U. Thus, for example, under $U \otimes U$, the space $L^2(M \times M)$ decomposes into two irreducible invariant subspaces, viz L^2 symmetric and L^2 antisymmetric. The corresponding unitary representations of G are inequivalent to each other and to U. One can also show that the natural action of G on the tangent bundle TM induces an irreducible representation on $L^2(TM)$.

B. Let (M, ω) be a symplectic manifold, and let $\mathfrak{F}$ be the algebra of C_0^∞ functions on M with Poisson bracket. Following van Hove [9] (see also [5, 6, 7, 8]) we define, for $f \in \mathfrak{F}$,

$$A(f) = X_f = \text{the Hamiltonian vector field of } f$$

so that $X_f \cdot g = \{f, g\}$. We then find that multipliers ρ correspond to derivations of $\mathfrak{F}$. Moreover $\mathfrak{F}$ has essentially one outer derivation θ with $\theta(1) = 1$. E.g. for $M = \mathbf{R}^{2n}$ with coordinates q, p, van Hove sets $\theta(f) \equiv f - \sum_{i=1}^{n} p_i \, \partial f / \partial p_i$. Finally, for each nonzero real λ, set $B_\lambda(f) = X_f + \sqrt{-1} \cdot \lambda \theta(f)$. Then, by Theorem 2, the representation B_λ acts irreducibly on $L^2(M, \omega^n)$. The representations B_λ are mutually inequivalent since they are different on the center ({scalars}) of $\mathfrak{F}$. One gets new inequivalent representations by reducing tensor powers of the B_λ's.

C. Now let (M, ω) be a **compact** symplectic manifold. Then, as Avez has shown, we get a derivation by setting

$$\theta(f) = \frac{1}{\mu(M)} \int_M f \, d\mu + \bar{f}, \quad \text{the mean of } f$$

where $\mu = \omega^n$. Theorem 1 establishes immediately Avez's theorem [**1**] that the corresponding representations B_λ are irreducible. Higher tensor powers yield new, inequivalent representations. Thus the question of their existence, raised in [**1**], has an easy positive answer.

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