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Random fields: analysis and synthesis, by Erik Vanmarcke, MIT Press, Cambridge, Mass., 1983, xii + 382 pp., $45.00. ISBN 0-262-22026-1

Spectral theory of random fields, by M. I. Yadrenko, Optimization Software, Inc., New York, N.Y., 1983, iii + 259 pp., $24.00. ISBN 0-911575-00-6

A stochastic process is a family of random variables (X_t) defined on some probability space. The parameter t assumes values in an index set T, and X_t assumes values in a measure space called the state space. Much of the theory developed originally in the cases where T is the set of positive integers or where T is the nonnegative real axis. In this context we think of X_t as a random function of time. While this theory was sufficient for the modeling of random functions measured over time, there arose the need for the consideration of random functions X_t where t assumes values in subsets of the plane or space. For example, weather variables recorded at various points in the atmosphere may be viewed as the realization of a stochastic process with $T = R^3$.

A *random field* is simply a stochastic process with T a measurable subset of R^d, for some $d \geqslant 1$. The pioneering works in the theory of stochastic processes of Lévy, Kolmogorov, Khintchine, Feller, and Doob were concerned almost exclusively with the case $d = 1$. But it was Lévy who first noticed the potential interest in the case $d \geqslant 1$. His early work on the Brownian motion of several parameters [**8**], now know as "Lévy's Brownian motion", pointed out the direction for much modern research in random fields. In recent years interest in random fields has penetrated the theory of martingales, Markov processes, Gaussian processes, additive processes, and general second-order stationary processes.

The branch of probability theory commonly called random fields, in particular, the material considered in the two books under review, has a scope more limited than that suggested by previous remarks. It is the subset of the theory dealing with extensions from $d = 1$ to $d \geqslant 1$ of the theory of processes that are stationary or have stationary increments, either in the strict sense or in the wide sense (second-order stationarity). Another recent book in this area is *The geometry of random fields* by R. J. Adler [**1**].

Second order random fields and isotropy. As is well known, (X_t), $t \in R^1$, is called second-order (wide sense) stationary if $EX_t = 0$ (or any constant) and $EX_sX_{t+s} = r(t)$, where the latter does not depend on s. If R^1 is extended to R^d, s and t belong to R^d, and if the condition stated above holds, then the random field is called homogeneous. The classical spectral theory in R^1 extends directly to R^d. This extension assumes significant interest when the assumption of *isotropy* is introduced: X_t is isotropic if the function $r(t)$ above depends on t only through the nonnegative variable $\|t\|$, where $\|\cdot\|$ is the usual R^d norm. The famous theorem of Schoenberg **[12]** states that the function $r(s)$ is necessarily of the form

$$r(s) = \int_0^\infty \frac{J_{(d-2)/2}(us)\, dG(u)}{(us)^{(d-2)/2}}, \qquad s \geqslant 0, \tag{1}$$

where G is a bounded nondecreasing function, and J_m is the Bessel function of the first kind of order m. This leads to the classical second-order expansion of X_t in spherical harmonics (H_v) on the unit sphere,

$$X_t = \sum_v c_v \xi_v(\|t\|) H_v(t/\|t\|), \tag{2}$$

where $(\xi_v(s),\ s \geqslant 0)$ are mutually orthogonal processes on R^1, and (c_v) is a sequence of constants. This theory extends to random fields with homogeneous and isotropic increments, where $E(X_t - X_s)^2$ is a function of $\|t - s\|$. Lévy initiated this work with the case $E(X_t - X_s)^2 = \|t - s\|$.

Yadrenko's book contains a comprehensive survey of this theory. The first half describes the spectral theory of isotropic fields, and the last quarter deals with the associated statistical questions of forecasting, extrapolation, and linear estimation. The remaining part of the book is concerned primarily with two specific problems in the special case where the field is Gaussian: the local smoothness of the sample functions, and the mutual absolute continuity and singularity of measures corresponding to two fields. Much of the book represents the author's own contributions over the years. It is written in the same style as the original work. It is a valuable collection of theoretical results and their proofs.

Distributions of functionals such as level sets, extremes, and sojourns in the Gaussian case. Now we discuss some of the portions of random field theory based on the extension of strict stationarity from $d = 1$ to $d \geqslant 1$. In both cases the class of processes that has been most extensively studied is Gaussian. The reason is that the problems that have arisen in applications, such as determining the distributions of functionals, require a fairly explicit representation of the finite-dimensional distributions of the process. This is one of the features of the Gaussian process in the stationary case. The use of Gaussian models in applications is, of course, justified by the same logic as its extensive use in classical statistics, namely, by the central limit theorem.

The pioneering work in the theory of functionals of stationary Gaussian processes was that of S. O. Rice **[11]** in 1945. This initiated four decades of activity that has increased in interest over the years. While most of this has been done in the case $d = 1$, there have also been extensions to $d \geqslant 1$. Consider the level set $(t\colon X_t = x)$, for fixed x. For $d = 1$ it represents the roots of a random equation in a real interval. The Rice Formula for the expected number of zeros of X_t was the first significant result, and much has been developed for $d = 1$ since that time; see, for example, Cramer and Leadbetter **[6]**. For $d > 1$ the study of the level set presents new and difficult challenges. Vanmarcke's book contains some of this material. A more comprehensive theoretical treatment is also given by Adler [**1**]. But Vanmarcke's book is different from the others in that it is intended as a self-contained text with just a calculus prerequisite, and includes introductory chapters on probability and stochastic processes. In addition to the results on the theory of functionals of random fields, it also contains an elementary introduction to the spectral theory of Yadrenko's book. Among its features are its organization into clearly marked chapters and subsections and its abundance of explicit formulas and calculations. It will be useful as a reference for applied probabilists and engineers, as well as a text.

The Markov property. The well-known Markov property of a stochastic process X_t, $t \in R^1$, is that for every τ, the past X_t, $t < \tau$, and the future X_t, $t > \tau$, are conditionally independent, given the present X_τ. It was apparently Lévy who first proposed the possibility of extending this concept to random fields. He defined Markovity in the following way: X_t, $t \in R^d$, is Markovian if for any surface S, which divides the space into two parts V_1 and V_2, the fields X_t, $t \in V_1$, and X_t, $t \in V_2$, are conditionally independent, given X_t, $t \in S$. However, it was soon discovered that the condition in this definition was so strong that the only Gaussian Markovian fields were those which were deterministic in the sense that their values assumed on any smooth surface in R^d determined their values over all of R^d. In particular, in the isotropic case this implied that $X_t = X_0$ almost surely for each t. Yadrenko's book describes some of this research.

It was H. P. McKean who found the right definition of Markovity for random Gaussian fields in 1963 [**9**]. Let the surface S be as above, and let U be an arbitrary neighborhood of S. Let $\mathscr{F}_U$ be the σ-field generated by X_t, $t \in U$, and put $\mathscr{F} = \bigcap_U \mathscr{F}_U$. The field is called Markovian if for any s and s' in the interiors of V_1 and V_2, respectively, X_s and $X_{s'}$ are conditionally independent, given $\mathscr{F}$. McKean proved that Lévy's Brownian motion over R^d was Markovian for odd d but not for even d. This led to Pitt's work [**10**] on the characterization of the spectral density of a homogeneous Gaussian field having the Markov property. A brief outline of these results, without proofs, is in Adler's book [**1**]. The definition of Markovity for fields was also specialized to the original case $d = 1$, where it has been of much interest in the context of "germ fields".

Hilbert space as the time domain ($d = \infty$). Some of the most striking results in this area are those for which the time domain is infinite dimensional. It was again Lévy who made the first significant contributions. He extended the Brownian motion of d parameters to Hilbert space by defining $E(X_t - X_s)^2 = \|t - s\|$, where $\|\cdot\|$ is the Hilbert norm. He showed that the sample paths were continuous on all finite-dimensional subspaces, but unbounded over all infinite-dimensional balls. This was complemented by the seemingly contradictory result that the values of X_t, for t in any ball, uniquely determine the value of X_s for every s in the space. Then he introduced the concept of the spherical average of X_t. Let B_n be the unit sphere in R^n, centered at the origin, and put

$$M_n = \int_{B_n} X_t \, dt / \mathrm{Area}(B_n).$$

Then the limit exists almost surely for $n \to \infty$ and is called the spherical average of X_t. Similarly, for each $t > 0$, $M(t) = \lim_n M_n(t)$ was defined as the spherical average over the sphere of radius t. Lévy was particularly interested in the process $M(t)$. Continuing this very imaginative work, he stated the theorem that X_t was a harmonic function over Hilbert space in the sense that the value of X_t was equal to the spherical average of the field over any sphere centered at t. However, the proof had an elementary error, and the result was never confirmed.

Other probabilists then considered the extensions of these results to more general Gaussian fields over Hilbert space. Yadrenko's book describes his work for isotropic Gaussian fields. The basic tool in this area is Schoenberg's representation of an isotropic covariance function on the sphere **[13]**:

$$(3) \qquad EX_sX_t = \sum_{m=0}^{\infty} c_m (s, t)^m,$$

where (c_m) satisfies $c_m \geqslant 0$, $\sum c_m = 1$, and (s, t) is the inner product. Much of Yadrenko's work is concerned with variations of the Markov property and their implications for the particular form of the covariance (3). The Russian edition of Yadrenko's book appeared in 1980, the year in which the reviewer's paper **[2]** was published. The latter contained several new results in the area of isotropic processes on the Hilbert sphere:

(a) The process is deterministic: Its values on an arbitrary open subset of the sphere determine its values over the whole sphere.

(b) Lévy's statement on the harmonicity of Brownian motion, which was never confirmed, is valid in a suitable form for a general isotropic process on the sphere: For any t on the sphere, X_t is representable as a series of averages of X_s for points s belonging to subspheres that are in the subspaces orthogonal to t.

(c) Let $f(x)$ be a function in $L_2(\phi(x)\,dx)$, where ϕ is the standard normal density, and define

$$M_n(f) = \int_{B_n} f(X_t)\,dt/\mathrm{Area}(B_n),$$

where B_n is the unit n-sphere. Then $M_n(f)$ converges almost surely to a random variable of an explicit form. Under further conditions on the coefficients in (3), $M_n(f)$ satisfies central limit theorems with either normal or nonnormal limiting distributions. The main tool in the proofs of these theorems is the representation of f in a Hermite polynomial expansion.

Finally, we remark that the theory has been extended from Hilbert space to l_p-space. Schoenberg showed **[12]** that the limiting form of (1) for $d \to \infty$ is

$$r(s) = \int_0^\infty e^{-u^2s^2}\,dG(u)$$

for some bounded nondecreasing G, so that isotropic covariance functions on Hilbert space are limited to this type. Bretagnolle, Dacunha-Castelle and

Krivine [4, 5] extended this to l_p. They showed that every isotropic covariance on the space is of the form

$$r(s) = \int_0^\infty e^{-u^p s^p}\, dG(u),$$

and they also studied the conditions for the determinism of the field.

Sets of continuity and discontinuity in infinite-dimensional spaces. Lévy's result on Brownian motion on Hilbert space–namely, that the sample functions are continuous on finite-dimensional subspaces but unbounded on infinite-dimensional balls–is typical of more general Gaussian fields having isotropic increments. The discontinuity is explained by the fact that the ball is not compact, so that there is sufficient "time" inside the ball or on its surface for the sample function to achieve a big jump. This raises the question of continuity on a compact subset of the space that is not contained in a finite-dimensional subspace, such as an ellipsoid. The question is resolved in the following way. Let C be the compact set under consideration, and define the metric $d(s, t) = (E(X_s - X_t)^2)^{1/2}$. Then X_t has continuous or bounded sample functions on C if the metric entropy of C, relative to the metric d, is sufficiently small, and, conversely, it has discontinuous or unbounded sample functions on C if the metric entropy is too large. The main reference here is the paper of Dudley [7] for subsets of Hilbert space.

Recently the reviewer [3] described sufficient conditions for the unboundedness of the sample functions of more general, nonGaussian fields over l_p. It was shown, by comparing the conditions with those for continuity of the field (see, for example, Weber [14]), that the results for an ellipsoid C are nearly the best possible ones in the general cases considered.

Conclusion. A concise historical description of the theory of random fields up to 1980, and the relevant literature is contained in Yadrenko's book in the section at the end with the title "Notes". Vanmarcke's book, which is written more with a view to applications, also contains many references to the applied literature in various fields of engineering.

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Modern geometry (*Sovremennaya geometriya*), by B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, Moskva:Izdatel'stvo "Nauka", 760 pp., R.2.00 (Russian), 1979.

This is a very important and interesting book. This volume is written by Russian authors who are experts in large areas of contemporary geometry and their applications. In particular, Novikov is a Field's Medalist and is publishing most exciting research in the intersection of geometry, analysis, and physics. The original Russian edition appeared in 1979 in a one-volume edition, double the size of the present translated volume. I first came across this book shortly thereafter when a new faculty member in our department, Alexander Eydeland, showed it to me. In 1982 at the colloquium in Paris to honor Laurent Schwartz, I found a translation of this book in French, in two volumes. The present English translation comprises the first volume of the French edition.

The best way to describe this volume is to say that it is a contemporary treatise on modern methods in geometry with deep applications to the physical sciences. It used to be that the great treatises of mathematics were written by scholars of analysis, and these volumes contained a synthesis of the geometry and analysis of their times. The latest example of this type of work is the comprehensive and marvelous treatise on analysis by Jean Dieudonné in twenty-five chapters, twenty-four of which have appeared in nine volumes.

As a student, I studied geometry for each of my undergraduate years, but unfortunately it was not the kind of geometry presented in this new Russian book. Indeed, it was filled with linear projective geometry in all its classical points of view (triangles, lines, hyperplanes, crossratios, and eventually conic sections). (Even Euclid would have felt maligned.) Eventually I escaped to England to discover that projective geometry could include cubic surfaces and,