volumes range from Feynman's three, to Arnold Sommerfeld's seven, to Landau's and Lifschitz's at least ten. Let us hope that this volume, with its incisive vision of the unity of mathematics, will initiate a similar fashion in the mathematical community. I believe such book writing is long overdue.

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Extremes and related properties of random sequences and processes, by M. R. Leadbetter, Georg Lindgren and Holger Rootzen, Springer Series in Statistics, Springer-Verlag, New York, Heidelberg, Berlin, 1983, xxi + 336 pp., $36.00. ISBN 0-387-90731-9

The classical theory of extreme values of probability theory deals with the asymptotic distribution theory of the maxima and the minima of independent and identically distributed (i.i.d.) random variables. That is, let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common distribution function $F(x)$. Put $W_n = \min(X_1, X_2, \ldots, X_n)$ and $Z_n = \max(X_1, X_2, \ldots, X_n)$. Then the distribution functions of W_n and Z_n satisfy

$$L_n(x) = P(W_n \leqslant x) = 1 - [1 - F(x)]^n$$

and

$$H_n(x) = P(Z_n \leqslant x) = F^n(x).$$

It is rare in probability theory that $F(x)$ is known. Indeed, the field of determining $F(x)$ from some elementary properties, known as characterizations of probability distributions, is quite recent (for the history of the field of characterizations, see the introduction in Galambos and Kotz (1978)). On the other hand, if $F(x)$ is determined by some approximation, however accurate, the values of $H_n(x)$ and $L_n(x)$ cannot be computed from the above formulas due to the sensitivity of u^n to u for large n (compare $0.995^{400} = 0.1347$ and $0.999^{400} = 0.6702$). This difficulty is overcome in an asymptotic theory that is invariant for large families of population distribution $F(x)$. In other words, for varying $F(x)$, linearly normalized extremes $(Z_n - a_n)/b_n$ or $(W_n - c_n)/d_n$ have the same limiting distribution functions $H(x)$ or $L(x)$, respectively. The theory is well developed for finding these appropriate normalizations and the forms of the limiting distribution functions, as well as for easy-to-apply criteria for $F(x)$ leading to a particular $H(x)$ or $L(x)$. Chapter 2 of Galambos (1978) gives a full account of this theory.

The classical theory of extremes can at best be applied as a first approximation to real-life models. Observations collected in, or produced by, nature are rarely independent, and neither are components of pieces of equipment functioning independently. For example, floods, defined as the highest (random) water level of a river at a given location, are clearly obtained through strongly dependent values, which dependence might weaken as time goes on. In engineering applications, on the other hand, when a piece fails as soon as one of its (major) components fails, and thus the life of the equipment is the minimum of the lives of its components, the dependence determined by the structure of the components might be so strong that the classical model could not even be used as a guide in determining the asymptotic distribution of the "life" of the equipment when the number n of components is large. A further need for the investigation of models other than the classical ones is the Bayesian thinking in statistics: If parameters are random, then independent observations at a given parameter value are, in fact, exchangeable random variables. These needs, as well as mathematical curiosities, led to the investigation of the asymptotic theory of extremes for a large variety of models. The following four models went through the most extensive development in the past two decades.

Stationary mixing sequences. Let $X_1, X_2, \ldots$ be identically distributed random variables, and let T be a finite set of positive integers. The X_j, $j \geqslant 1$, are said to be stationary if, for every integer $s \geqslant 1$, the joint distribution of $\{X_j, j \in T\}$ is the same as that of $\{X_{j+s}, j \in T\}$. Furthermore, this sequence is mixing if the previous two blocks of random variables are asymptotically independent as $s \to +\infty$. Loynes (1965) gave a detailed analysis of the asymptotic distribution of the extremes for this model. Supplemented by the results of O'Brien (1974), the conclusion of Loynes is that the asymptotic behavior of the extremes in a stationary mixing sequence, under some additional assumption, is similar to the classical case.

E_n^*-sequences. Let $G = (V, E)$ be a graph with vertex-set $V = \{1, 2, \ldots, n\}$ and $E \subset V^2$ (the edges of the graph). The vague description of an E_n^*-sequence is as follows (for an exact definition see pp. 176–177 of Galambos (1978)). The random variables $X_1, X_2, \ldots, X_n$ form an E_n^*-sequence if the events $\{X_j \geqslant z_n\}$ satisfy the following dependence requirements. Here, z_n is a sequence of numbers such that $F_j(z_n) < 1$ for all j and $\sup_j F_j(z_n) \to 1$ as $n \to +\infty$. First, if $(i_1, i_2, \ldots, i_k)$ is such that no pair (i_j, i_m) belong to E, then the events $\{X_{i_j} \geqslant z_n\}$ are asymptotically independent. Second, if exactly one pair $(i_j, i_m) \in E$, then the probability of the joint occurrence of $\{X_{i_t} \geqslant z_n\}$, $1 \leqslant t \leqslant k$, can be majorized by the product of $P(X_{i_j} \geqslant z_n, X_{i_m} \geqslant z_n)$ and $P(X_{i_t} \geqslant z_n)$, $t \neq j, m$. Finally, the number of elements of E is $o(n^2)$. In this model, under some conditions, the asymptotic distribution of extremes, when normalized, is the same as if they were independent. Consequently, all distributions with monotonic hazard rate are possible limiting distributions. This is pleasing in engineering applications where distributions with monotonic hazard rate are widely applied as life distributions without reference to extreme value theory. This model gives the necessary theoretical justifications to such applications.

It should be noted that the present model contains as a special case the stationary mixing sequences. Simply take an integer $s = s(n) = o(n)$. Let E be defined as the set (i, j) with $|i - j| < s$. Then the conditions in this model are somewhat less restrictive than those of Loynes, even when stationarity is added to the assumptions.

Gaussian sequences. If the joint distribution of $X_1, X_2, \ldots, X_n$ is multivariate normal, we speak of Gaussian sequences. If the sequence X_j, $j \geqslant 1$, is stationary Gaussian, and if the correlation coefficient r_m of X_j and X_{j+m} satisfies

$r_m \log m \to 0$ as $m \to +\infty$, then the result of Berman (1964) says that the asymptotic distribution of the extremes is the same as if they came from an i.i.d. sequence of normal variables. When the assumption on the correlation coefficient fails, then new asymptotic laws are obtained for the normalized extremes: Among the possibilities are the normal distribution and its mixtures with the classical extreme-value distributions (Mittal and Ylvisaker (1975)).

Exchangeability. It was mentioned earlier that exchangeable variables are as basic in Bayesian statistics as independence is for non-Bayesian statistics. Purely probabilistic arguments can also lead to exchangeable variables. The asymptotic theory of extremes for these models was developed by Berman (1962) and Galambos (1973) and (1975). The theory is significantly different, depending on whether the basic random variables constitute a finite segment of an infinite sequence of exchangeable variables, or whether they come from a finite sequence which cannot be further extended without violating exchangeability. In both cases, however, the limiting distributions of the extremes are mixtures of the classical limiting distributions with an arbitrary distribution. When this latter mixing distribution is degenerated at a point, the classical limiting distributions are reobtained. Criteria for this to happen are also available.

These four models, together with the classical one, are fully covered in the book by Galambos (1978). The book under review concentrates on the model of stationary mixing sequences, with a short review of the classical model and the Gaussian case. No mention is made that other models have also been investigated, although the list of references from Galambos (1978) is reproduced (without acknowledgement), so the interested reader can find out what else had been done prior to 1978.

The book under review also discusses extreme values in continuous time. It presents a detailed development for the case of stationary normal processes for which several explicit extremal results are known. The extremal theory is approached through the consideration of properties of upcrossings of a high level. When normality is not assumed, the extremal theory of stationary continuous time processes is reduced to discrete parameter results. Although the discrete parameter case is very restrictive, and thus it does not represent current knowledge in the field, the corresponding dependence assumptions in the continuous time case lead to one of the most general results known today.

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Unified integration, by E. J. McShane, Academic Press, Orlando, Fl., 1983, xii + 607 pp., $55.00. ISBN 0-12-486260-8

This book is intended to provide a unified theory of integration (Riemann, Lebesgue, etc). Professor McShane believes that the way integration is taught now is not very efficient, since we first teach our students the Riemann integral, and then, once we have introduced them to the Lebesgue integral, abandon all our earlier work about the Riemann integral.

In this book Professor McShane tries to produce a unified way of defining the integral; of course, he includes, as a special case, the Riemann integral. The author hopes that this way of introducing the integral "can also go from beginning calculus to the graduate level without ever abandoning earlier work and starting again (as usually now happens when Lebesgue integration is met)".

The book under review begins with the introduction of the gauge integral. Some preliminaries are necessary for its definition.

An *allotted partition* of a set B in R is a finite set of pairs

$$P = \{(x_1, A_1), \dots, (x_k, A_k)\}$$

in which the A_i are pairwise disjoint left open intervals in R, the x_i are points in $[-\infty, +\infty]$, and $B = \bigcup_{i=1}^k A_i$.

To an allotted partition P and an extended real-valued function f, $S(P, f)$ will denote the sum $\sum_{i=1}^k f(x_i)m(A_i)$, where $m(A_i)$ is the length of the interval A_i.

A *gauge* Γ on a set B in $[-\infty, +\infty]$ is a function $x \to \Gamma(x)$ such that, for each x in B, $\Gamma(x)$ is a neighborhood of x.

If Γ is a gauge on $[-\infty, +\infty]$ and P is an allotted partition of a set B that is equal to the union of the A_i's, P is said to be Γ *fine* if, for each $i = 1, 2, \dots, k$, the closure of A_i is included in $\Gamma(x_i)$.

Now, the definition of the gauge integral is given as follows:

DEFINITION. Let B be a subset of R and f a real-valued function defined on a subset D of $[-\infty, +\infty]$. Suppose B is contained in D. Define g to be equal to f on B and to be zero on $[-\infty, +\infty] \setminus B$. The function f is said to be *gauge integrable* on B and of gauge integral J if for every $\varepsilon > 0$ there exists a gauge Γ on $[-\infty, +\infty]$ such that if P is a Γ fine partition of R, $S(P, g)$ exists and $|S(P, g) - J| < \varepsilon$.