on intervals and "was assumed only to be finitely additive and nonnegative, and it was that proof of the monotone convergence theorem that brought countable additivity into the theory ". The point the reviewer would like to make is that starting from an elementary definition does not always protect someone from paying the price later on in the theory.

This book is very well written, contains some very good exercises, and proofs are given in full detail. It is an honest attempt by somebody who loves measure theory to try to make this very important tool (the Lebesgue integral) accessible to a wide audience.

How well this book will succeed in achieving its avowed purpose of making the unified treatment of integration widely accepted is perhaps better judged by how fast and how often this book, or similar books, will make it to the classroom.

ELIAS SAAB

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*Group representations and special functions*, by Antoni Wawrzyńczyk, Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht/Boston/Lancaster, 1984, xvi + 687 pp., $119.00 US; Dfl. 320,--. ISBN 90-277-1269-7

This huge book, a translation of the 1978 Polish original [1], is clearly intended by the author to be a study of the relations between the representation theory of groups and the special functions of mathematical physics. What has emerged is somewhat more restricted: a detailed and extensive study of the theory of spherical functions and harmonic analysis on symmetric spaces, and the application of these theories to certain special functions. The so-called special functions of mathematical physics are those useful functions which arose when physicists obtained explicit solutions of the partial differential equations governing physical phenomena—e.g., the heat, wave, Helmholtz, and Schrödinger equations—through separation of variables. With the development of quantum mechanics in the 1920s and 1930s, it became evident that there were relations between the symmetries of the partial differential equations and some of the special functions that arose as solutions of these equations. However, the first clear formulation of such a relationship appears in Eugene Wigner's 1955 unpublished Princeton lecture notes. The first extensive published treatment of the theory was the 1965 monograph of N. J. Vilenkin in which the achievements of the Gel'fand school in the theory of spherical functions were utilized [2]. This was followed by J. D. Talman's book in 1968, based on Wigner's lectures [3]. In these works the special functions occur as matrix elements of irreducible representations of the fundamental symmetry groups of physics. The matrix elements are defined with respect to a basis which, typically, is chosen to have simple transformation properties under the action of some subgroup, such as a maximal compact subgroup. Thus, Bessel functions appear as matrix elements of representations of the Euclidean motion group in two-space. Jacobi and Legendre polynomials are associated with representations of SU(2), Gegenbauer polynomials arise from representations of SO($n$), Jacobi functions are associated with SL(2, $R$), and Laguerre polynomials arise as matrix elements of representations of the Heisenberg group.

Some of the basic properties of these functions follow easily from this relationship. Thus, the group multiplication rule for the matrix elements leads to addition theorems for the special functions. The Peter-Weyl theorem and the orthogonality relations for matrix elements of irreducible group representations lead to integral relations and completeness theorems for special functions. By passing from the group to the corresponding Lie algebra, one obtains differential equations and differential recurrence relations for the matrix elements. The natural abstract setting for all these results is the theory of spherical functions and harmonic analysis on symmetric spaces, which has a beautiful presentation in the textbooks of Helgason [**4, 5**].

Work on the interpretation of special functions as spherical functions by no means ended with the books of Vilenkin and Talman, however. For instance, there is the reviewer's book [**6**] in which local Lie group methods are used to generalize the classes of special functions that can be treated via group theory. In this work it is shown that the factorization method of Inoui [**7**] and of Infeld and Hull [**8**] for solving eigenvalue problems associated with the Schrödinger equation has a Lie algebraic interpretation. Koornwinder [**9–11**] used the theory of spherical functions, at least in part, to establish a new general addition theorem for Jacobi polynomials (extending the classical addition theorem for Gegenbauer polynomials). It is known that all spherical functions on symmetric spaces can be interpreted as families of orthogonal polynomials. Askey and Wilson [**12**] reinterpreted the Racah coefficients for representations of SU(2) as families of orthogonal polynomials and then extended these to find a general set which contains, strictly, all the spherical functions on symmetric spaces. There has been considerable progress in the study of special functions that arise as spherical functions associated with finite groups, particularly in the context of 2-point homogeneous spaces. For example, the Hahn and Krawtchouk polynomials, classical discrete orthogonal polynomials, occur in this way. Furthermore, for Chevalley groups over the finite field GF($q$) one obtains orthogonal polynomials given by basic hypergeometric series ($q$-series). See the paper by Stanton [**13**] for an excellent survey.

As mentioned previously, the special functions of mathematical physics arise via separation of variables in the partial differential equations of physics. It is simply not the case that all these functions are spherical functions (or that they arise as matrix elements of group representations). For example, the functions of Mathieu, Lamé, and Ince, ellipsoidal wave functions, products of parabolic cylinder functions, and many multivariable hypergeometric functions are not spherical functions even though they arise via separation of variables. In the reviewer's book [**14**], based on work carried out jointly with E. G. Kalnins, it is shown that these functions can be characterized as eigenfunctions of a complete set of commuting operators that are second-order elements in the enveloping algebra of the Lie algebra associated with the symmetry group of the original partial differential equation (or system of equations). Thus, in effect, all of the standard functions of mathematical physics can be treated via group representation methods, but many of them do not yield to the methods presented in the book under review.

With this very sketchy survey of recent work out of the way, we turn to the book itself. It is divided into three approximately equal parts. In the first part the author develops from first principles the basic facts relating Lie groups and Lie algebras, homogeneous spaces, the representation theory of locally compact groups, direct integrals of group representations, decomposition theory of unitary representations, representations of compact groups (including the Frobenius theorem and the Peter-Weyl theory), and the theory of spherical functions. In Part II the theory of spherical functions is applied to obtain specific facts about special functions. The special functions treated, all obtained as matrix elements, are the gamma function, Bessel functions, Jacobi, Legendre, Gegenbauer, and Laguerre polynomials, as well as Jacobi and Legendre functions. Attention is paid to harmonic analysis involving these functions. As the author states, Part II is a modification of Vilenkin's monograph **[2]**. Part III is based on Helgason's monograph **[15]** and contains an introduction to the geometry of general symmetric spaces and to harmonic analysis on these spaces. It repeats some of the material in the earlier sections of the book, but at a higher level. Such topics as affine transformations, Riemannian symmetric spaces, the structure theory of semisimple Lie algebras, the Harish-Chandra $c$-function, the Radon transformation, and the Paley-Wiener theorem are treated.

The presentation is clear, and many examples and exercises are provided. Typographical errors are numerous but mostly harmless. (The most spectacular of these is the information about Klein's Erlangen program of 1972!) The author covers an immense amount of material at the expense of stating dozens of important theorems without proof (and occasionally without a reference). Since even such basic results as the connection between a Lie group and its Lie algebra are stated without proof, some readers may be frustrated. (Many theorems are proved, however). The material in Parts I and III is presented at a high level of generality, and many theorems are quoted which are not needed for application to special functions in Part II.

This book is based heavily on the monographs of Vilenkin **[2]** and Helgason **[15]** and is unique in the sense that it intertwines these two approaches. (Remarkably, Helgason's book **[5]** is not referenced.) With the exception of two sections on the Radon transform, adapted from Helgason, virtually all results date from prior to 1970, although the references at the end of the book do contain later work. The reader is never told that there is a difference between spherical functions and special functions. Koornwinder's addition theorem and Koornwinder, himself, are not mentioned, nor are spherical functions associated with finite groups.

In conclusion, the author's presentation is attractive and lucid, quite suitable for a graduate level course on spherical functions with applications to special functions. However, for a modern unified approach to special functions based on group theory, one should look elsewhere.

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*The umbral calculus*, by Steven Roman, Academic Press, Orlando, Fl., 1983, x + 193 pp., $35.00. ISBN 0-12-594380-6

The umbral, or symbolic, notation was originated by Aronhold and Clebsch in the middle of the nineteenth century and proved to be an important tool in the theory of algebraic invariants. It was later taken on by Blissard, who applied it to derive various algebraic and combinatorial identities.

The idea behind the umbral notation is to start with an algebraic identity involving powers $\{a^k\}$, $\{b^k\}$ and replace $a^k \leftarrow a_k$, $b^k \leftarrow b_k$, where $\{a_k\}$, $\{b_k\}$