In conclusion, the author's presentation is attractive and lucid, quite suitable for a graduate level course on spherical functions with applications to special functions. However, for a modern unified approach to special functions based on group theory, one should look elsewhere.

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The umbral calculus, by Steven Roman, Academic Press, Orlando, Fl., 1983, x + 193 pp., $35.00. ISBN 0-12-594380-6

The umbral, or symbolic, notation was originated by Aronhold and Clebsch in the middle of the nineteenth century and proved to be an important tool in the theory of algebraic invariants. It was later taken on by Blissard, who applied it to derive various algebraic and combinatorial identities.

The idea behind the umbral notation is to start with an algebraic identity involving powers $\{a^k\}$, $\{b^k\}$ and replace $a^k \leftarrow a_k$, $b^k \leftarrow b_k$, where $\{a_k\}$, $\{b_k\}$

are now *arbitrary* sequences. Perhaps the simplest use of the umbral method is to derive the following pair of inverse relations:

$$(1) \qquad a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \qquad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

Let $a = b + 1$; then $b = a - 1$, and we have

$$a^n = (b+1)^n, \qquad b^n = (a-1)^n.$$

Thus

$$a^n = \sum \binom{n}{k} b^k, \qquad b^n = \sum (-1)^{n-k} \binom{n}{k} a^k.$$

Now replace $a^n \leftarrow a_n$, $b^n \leftarrow b_n$ and get (1).

Although the symbolic notation proved to be very effective, there were many doubts as to its legitimacy, and Cayley, for example, had to resort to differential operators and "hyperdeterminants" to convince himself of its validity. Much later, E. T. Bell, in 1941, proposed a rigorous foundation to the umbral calculus which, though valid, was rather contrived. It was only in the early 1970s that Gian-Carlo Rota came up with a truly convincing explanation of the umbral calculus. We will illustrate it by rederiving (1) the Rota way. Introduce linear operators A, B on the vector space of polynomials $p(z)$ defined on the basis $\{z^n\}$ by $A(z^n) = a_n$, $B(z^n) = b_n$. Then

$$a_n = \sum \binom{n}{k} b_k \Rightarrow A(z^n) = \sum \binom{n}{k} B(z^k) = B\left(\sum \binom{n}{k} z^k\right) = B((z+1)^n).$$

By linearity, for every polynomial $p(z)$,

$$A(p(z)) = B(p(z+1)).$$

Thus

$$B(p(z)) = A(p(z-1)),$$

and, in particular,

$$\begin{aligned} b_n = B(z^n) = A((z-1)^n) &= A\left(\sum (-1)^{n-k} \binom{n}{k} z^k\right) \\ &= \sum (-1)^{n-k} \binom{n}{k} A(z^k) = \sum (-1)^{n-k} \binom{n}{k} a_k. \end{aligned}$$

A sequence of polynomials $\{p_n(x)\}$ is of *binomial type* if it satisfies a "binomial theorem"

$$p_n(x+y) = \sum \binom{n}{k} p_k(x) p_{n-k}(y).$$

Given a sequence of binomial type $p_n(x)$, a sequence of polynomials $\{s_n(x)\}$ is said to be *Sheffer* with respect to $\{p_n(x)\}$ if

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) s_{n-k}(x).$$

The Rota-Roman umbral calculus, the subject matter of Steve Roman's book, is a new approach to the theory of Sheffer sequences. Its novelty is in that, unlike Sheffer's analytical treatment, it is purely algebraic, residing on the notion of the umbra—that is, of linear functionals on the vector space of polynomials. To describe their approach we need some more of their notation.

There is a 1-1 correspondence between linear functionals on the vector space of polynomials, defined on the basis by $L(z^n) = a_n$ and formal power series $f(t) = \Sigma a_n t^n/n!$. Roman actually identifies the two notions and represents linear functionals by their formal power series. Roman and Rota's definition of Sheffer sequences is surprisingly simple and elegant. Let $f(t)$ and $g(t)$ be linear functionals such that $f(0) = 0$, $g(0) \neq 0$. Then the Sheffer sequence $\{s_n(x)\}$ corresponding to (f, g) is given by the orthogonality condition

$$g(t)f(t)^k(s_n(z)) = n!\delta_{n,k}.$$

Starting with this definition, Rota and Roman show that all other properties of Sheffer sequences follow. In particular, the generating function

$$\sum_{k=0}^{\infty} \frac{s_k(y)}{k!}t^k = \frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)},$$

where $\bar{f}(t)$ is the inverse power series of $f(t)$: $f(\bar{f}(t)) = t$.

Steve Roman's book is an exceptionally clear and self-contained exposition of this elegant theory. The first three chapters deal with the theory itself. The fourth chapter gives many interesting examples that nicely illustrate the wide scope of the theory, and Chapter 5 has "topics". Two of these topics are particularly noteworthy. §5.1 gives a full treatment of how to determine the "connection coefficients" $c_{n,k}$ of two given Sheffer sequences $\{s_n\}$, $\{r_n\}$,

$$s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x),$$

and §5.5 gives a nice account of "inverse relations" (the most simple example of which was described above). It is amazing that the umbral calculus can handle all the inverse relations in Riordan's [4] famous book.

The final chapter, Chapter 6, deals with nonclassical umbral calculi which are analogues for which $\{n!\}$ is replaced by other sequences.

My only reservation about this charming little book is that it almost solely concentrates on research in which the author himself took part. Such important work as Joni's [3] multivariate Lagrange inversion is only listed in the bibliography, and the significant contributions of the Italian school [1] did not even make it to the bibliography.

To conclude this review let me try and put the umbral calculus in historical perspective. Euler and his contemporaries manipulated power series "formally" without worrying much about convergence. This changed when Cauchy and Weierstrass developed rigorous notions of convergence, and, ever after, mathematicians were very uptight about convergence. Only relatively recently has it been realized that, in many cases, Euler was right after all, and that most manipulations in power series can be thought of as taking place in the (perfectly rigorous) algebra of formal power series. Even today this fact is not fully realized, and many authors still pay lip service to convergence. The umbral calculus, being purely algebraic, was a major force in shattering the idol of convergence. Ironically, the new insight that it afforded us provides us with the hindsight to realize that, by now, it is partly superfluous, and that many of its results can be proved directly in the framework of formal power series without the intervention of "umbra". For example, Chapter 3 culminates with a theorem that is equivalent to the famed Lagrange inversion formula. The Lagrange inversion formula, traditionally belonging to analysis, is now fully realized to be a purely algebraic fact, and a very short algebraic proof can be found, for example, in Hofbauer **[2]**.

But even if it is true, as some people claim, that everything that the umbral calculus can do can be done faster with just formal power series, nobody can deny the elegance, insight, and sheer beauty that the umbral calculus possesses, and Steve Roman's book is an excellent account of this beautiful theory.

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Theory of function spaces, by Hans Triebel, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983, 284 pp., $34.95. ISBN 3-7643-1381-1

The paradox of Besov spaces is that the very thing that makes them so successful also makes them very difficult to present and to learn. The idea behind Besov spaces begins with a simple extension of the idea of Lipschitz continuity, augmented by the observation that higher-order differences must also be used. For $s > 0$ choose any integer k greater than s. Differences are defined inductively. The first difference of a function is $f(x+h) - f(x)$ $(x \in R^n)$, and the kth difference is the composition of the $(k-1)$st and the first difference and is denoted Δ_h^k. Let $F(x, h) = \Delta_h^k f(x)/|h|^s$. The Besov space norm of f is the L^p norm of F in x followed by the L^q norm in h with respect to the measure $dh/|h|^n$. A function is in B_{pq}^s if f is in L^p and its Besov space norm is finite.

It was quickly found that there were many alternative approaches which give these same spaces. Today it is known that spaces defined by degree of approximation by entire analytic functions, spaces of functions which are values at 0 of solutions of the heat equation or Laplace's equation subject to