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Probabilistic and statistical aspects of quantum theory, by A. S. Holevo, North-Holland Series in Statistics and Probability, Vol. 1, North-Holland Publishing Company, Amsterdam, 1982, xii + 312 pp., $85.00, Dfl. 225.00. ISBN 0-444-86333-8

It has been known for over sixty years that quantum mechanics is, by its very nature, a statistical theory. The predictions of quantum mechanics are probabilistic and cannot be exact. However, the probability theory underlying quantum mechanics is not classical probability theory. It is a different kind of probability theory, which is phrased in terms of operators on a Hilbert space. Operators play the role of probability measures and random variables in quantum probability theory. Since these operators need not commute, quantum probability is sometimes called a noncommutative probability theory. This noncommutativity is the main difference between the two theories. The present book gives an account of recent progress in the statistical theory of quantum measurement stimulated by new applications of quantum mechanics, particularly in quantum optics. The main stress of the book is on the recently developed field of quantum estimation.

Quantum probability theory is attracting the attention of an increasing number of researchers. It is located at a junction between physics (in particular, quantum mechanics) and mathematics. It combines a blend of the abstract and the practical. It has been investigated by philosphers, physicists, mathematicians, electrical engineers, and computer scientists. There are interesting lines of research in this field for mathematicians who know little about quantum mechanics. For the mathematician there is a fascinating interplay between probability theory and functional analysis. There are also applications in this field involving group representations, operator algebras, partial differential equations, Schwartz distributions, functional integration, lattices, algebraic logic, and many others. This field was first presented in a rigorous operator-theoretic setting by von Neumann **[9]**. Later books which emphasize its geometric and logical aspects are **[1, 4, 5, 6, 8]**, and still more recent books in which probabilistic methods are treated are **[2, 3, 7]**. The present book is the first to emphasize the quantum estimation problem.

An example in which quantum estimation is required is in the field of optical communication, such as when laser beams are transmitted along glass fibers. In ordinary communication theory the signal of a radio wave, for example, is distorted by thermal noise in the air, and traditional statistical signal estimation can be used. However, in the optical range "quantum noise" becomes more significant than thermal background radiation in distorting the signal. In this case quantum estimation theory must be used. In particular, so-called Gaussian states are used to describe radiation fields in optical communication theory. The book derives general inequalities for mean-square measurement errors which are quantum analogs of the well-known Cramer-Rao inequality in statistics.

We shall now present some of the basic ingredients of quantum probability theory in the flavor of this book. In any experiment one can identify two main stages. The first stage is the preparation in which the initial conditions or input data of the experiment are established. In the second stage the prepared object is coupled to a measuring device, resulting in output data. We call the elements of the first stage *states*, and the elements of the second stage *measurements*. Denote the set of discernible states by $\mathscr{S}$ and the set of measurements by $\mathscr{M}$. Assume for simplicity that the values of a measurement $M \in \mathscr{M}$ are real numbers, and let $\mathscr{B}(\mathbb{R})$ be the σ-algebra of Borel subsets of $\mathbb{R}$. Now, suppose the object is prepared in the state S many times, and each time a measurement M is made. We would then obtain a statistical distribution of values for M given by a probability measure μ_S^M on $\mathscr{B}(\mathbb{R})$. Thus, we can think of M as a map from $\mathscr{S}$ to the set of probability measures on $\mathscr{B}(\mathbb{R})$.

If S_1, S_2 are states and $0 < \lambda < 1$, then it is usually not difficult to construct a state $S_3 = \lambda S_1 + (1 - \lambda)S_2$. The state S_3 would satisfy

$$\mu_{S_3}^M = \lambda\mu_{S_1}^M + (1 - \lambda)\mu_{S_2}^M$$

for every $M \in \mathscr{M}$. These considerations motivate the following definition. A *statistical model* is a pair $(\mathscr{S}, \mathscr{M})$ where $\mathscr{S}$ is a convex set and $\mathscr{M}$ is a set of affine maps from $\mathscr{S}$ into the convex set of probability measures on a fixed measurable space.

The statistical model used for classical probability theory is straightforward. Let (Ω, Σ) be a measurable space. Then $\mathscr{S}$ is the set of probability measures on Σ and $\mathscr{M}$ is the set of real-valued measurable functions (random variables) on Ω. For $S \in \mathscr{S}$, $M \in \mathscr{M}$ we obtain the probability distribution

$$\mu_S^M(B) = S\left[M^{-1}(B)\right], \qquad B \in \mathscr{B}(\mathbb{R}).$$

In this way $(\mathscr{S}, \mathscr{M})$ gives the classical statistical model.

The statistical model for quantum mechanics is quite a different story. In this case we have a complex Hilbert space H, and the states $\mathscr{S} = \mathscr{S}(H)$ are described by positive trace class operators of trace 1. (This can be "derived", more or less, from physical principles.) Such operators are called *density operators* on H. In order to describe quantum measurements we need a definition. A *resolution of the identity* on H is a set $M = \{M(B) \colon B \in \mathscr{B}(\mathbb{R})\}$ of bounded operators on H satisfying

(1) $M(\varnothing) = 0$, $M(\mathbb{R}) = I$,

(2) $M(B) \geqslant 0$, $B \in \mathscr{B}(\mathbf{R})$,

(3) if $B_i \in \mathscr{B}(\mathbf{R})$, $i = 1, 2, \ldots$, are mutually disjoint, then $M(\bigcup B_i) = \sum M(B_i)$ in the weak operator topology.

We say that M is an *orthogonal* resolution of the identity if each $M(B)$, $B \in \mathscr{B}(\mathbf{R})$, is a projection. A resolution of the identity is sometimes called a POV (*positive operator-valued*) *measure*, and an orthogonal resolution of the identity is sometimes called a PV (*projection-valued*) measure. Now define a *measurement* to be an affine map from the convex set $\mathscr{S}(H)$ to the convex set of probability measures on $\mathscr{B}(\mathbf{R})$. It is proved that $S \to \mu_S$ is a measurement if and only if there exists a unique resolution of the identity M such that $\mu_S(B) = \operatorname{tr} SM(B)$ for all $B \in \mathscr{B}(\mathbf{R})$. We thus take for the statistical model of quantum mechanics a pair $(\mathscr{S}, \mathscr{M})$, where $\mathscr{S} = \mathscr{S}(H)$ for a Hilbert space H and $\mathscr{M}$ is the set of resolutions of the identity for H. Moreover, for $S \in \mathscr{S}$, $M \in \mathscr{M}$, we have $\mu_S^M(B) = \operatorname{tr} SM(B)$ for every $B \in \mathscr{B}(\mathbf{R})$.

Measurements described by PV measures are called *simple*. Simple measurements are extreme points of the convex set $\mathscr{M}$ of all measurements; the converse is not true. As is well known from the spectral theorem, simple measurements are uniquely described by selfadjoint operators. Such operators are frequently thought of as representing all the observables of a quantum system. However, it is stressed in this book that there are important measurements (observables) which are not simple. For example, densely defined, symmetric (but not selfadjoint) operators are sometimes encountered in quantum mechanics (a specific case is the time observable). As shown in the book, these operators correspond to (in general, nonunique) POV measures in much the same way as selfadjoint operators correspond to unique PV measures. Moreover, using Naimark's theorem, any measurement can be dilated to a simple measurement in a larger Hilbert space.

The *expectation* and *variance* of a measurement M in the state S are defined by

$$E_S(M) = \int \lambda\, \mu_S^M(d\lambda), \qquad D_S(M) = \int [\lambda - E_S(M)]^2\, \mu_S^M(d\lambda).$$

In particular, if S is a one-dimensional projection P_ψ onto a unit vector ψ (*pure state*) and M is a POV measure for a densely defined symmetric operator X, with $\psi \in \mathscr{D}(X)$, then

$$E_S(M) = E_\psi(X) = \langle X\psi, \psi \rangle, \qquad D_S(M) = D_\psi(X) = \|X\psi\|^2 - E_\psi(X)^2.$$

For a pair of observables X_1, X_2 and $\psi \in \mathscr{D}(X_1) \cap \mathscr{D}(X_2)$, one easily obtains the *uncertainty relation*

$$D_\psi(X_1) D_\psi(X_2) \geqslant |\operatorname{Im} \langle X_1\psi, X_2\psi \rangle|^2.$$

An important role in quantum mechanics is played by symmetry groups. Let G be a group of symmetries for a quantum system. For example, G might be the Galilean group of rotations and translations or the relativistic Poincaré group of space-time. If $(\mathscr{S}, \mathscr{M})$ is a statistical model, then G must act on $\mathscr{S}$ and $\mathscr{M}$, and if G is really a symmetry of the system, we have $\mu_{gS}^{gM} = \mu_S^M$ for every $S \in \mathcal{S}$, $M \in \mathcal{M}$, $g \in G$. Moreover, it is natural to assume that $S \to gS$ is an affine bijection (or automorphism) on $\mathcal{S}$. If $\mathcal{S} = \mathcal{S}(H)$, then under mild continuity conditions it is proved that there exist unitary (or anti-unitary) operators V_g on H for each $g \in G$ such that

$$V_{g_1}V_{g_2} = \omega(g_1, g_2)V_{g_1g_2}, \quad \text{where } \omega(g_1, g_2) \in \mathbf{C}, \ |\omega(g_1, g_2)| = 1,$$

and

$$gS = V_gSV_g^* \quad \text{for all } g \in G, S \in \mathcal{S}.$$

Such a map $g \to V_g$ is called a *projective unitary representation*. In a similar way we have

$$gM(B) = V_g^*M(B)V_g \quad \text{for all } M \in \mathcal{M}, g \in G, B \in \mathcal{B}(\mathbf{R}).$$

If M is a POV measure from a σ-algebra $\mathcal{A}(U)$ of subsets of U, and G is a group of transformations acting transitively on U, we say that M is a *covariant measurement* if

$$M(g^{-1}B) = V_g^*M(B)V_g \quad \text{for all } g \in G, B \in \mathcal{A}(U).$$

Now, suppose that $\theta \to \tilde{V}_\theta$ is a projective unitary representation of the additive group $\mathbf{R}$. It can be shown that $\tilde{V}_\theta$ is equivalent to a unitary representation $\theta \to V_\theta$ on the same Hilbert space H. By Stone's theorem there exists a selfadjoint operator A on H such that $V_\theta = e^{i\theta A}$, $\theta \in \mathbf{R}$. Hence, the action on the states becomes

$$S \to S_\theta = e^{i\theta A}Se^{-i\theta A}.$$

Let $S = P_\psi$ be a pure state with $\psi \in \mathcal{D}(A)$. Then

$$\psi_\theta = e^{i\theta A}\psi \in \mathcal{D}(A),$$

and we write $E_\theta(X) = \langle X\psi_\theta, \psi_\theta \rangle$ for an observable X with $\psi_\theta \in \mathcal{D}(X)$. Using the uncertainty relation, we obtain the Mandelstam-Tamm inequality

$$D_\theta(X)D_\theta(A) \geqslant \tfrac{1}{4}|dE_\theta(X)/d\theta|^2.$$

This inequality is important for the quantum estimation problem. Suppose a quantum system is prepared in the initial state S which is completely known. Then the system is transformed according to a change of parameter θ, and the new state becomes S_θ. The value θ is supposed to be unknown, and the problem is to estimate this value statistically by making a measurement of the system. Suppose a measurement is made for the observable X. We then call X a *statistical estimate* of the parameter θ. (If the unknown parameter θ has a finite number of values, one speaks of "hypothesis testing" in statistics.) The accuracy of the estimate can be measured by the mean-square deviation

$$E_\theta\left[(X - \theta)^2\right] = D_\theta(X) + [E_\theta(X) - \theta]^2.$$

Then the Mandelstam-Tamm inequality sets a lower bound

$$E_\theta\left[(X - \theta)^2\right] \geqslant b(\theta)^2 + [1 + b'(\theta)]^2/4D_\theta(A),$$

where $b(\theta) = E_\theta(X) - \theta$ is the *bias* of the estimate X.

The set of estimates we have just considered is much too general. We want a statistical estimate for the parameter θ, but the observable X which we used may have nothing to do with θ. We must require some properties for X which relate it to the parameter θ. One way of doing this is the following: An estimate X is called *unbiased* if $b(\theta) = 0$ or $E_\theta(X) = \theta$, $\theta \in \mathbb{R}$. This means that there is no systematic error in the measurement. For unbiased estimates the above inequality takes the simple form $D_\theta(X) \geqslant [4D_\theta(A)]^{-1}$. One can also look for optimal measurements having the best theoretically possible accuracy among all covariant measurements of θ. It turns out that such measurements correspond to "canonical" observables for the parameter θ. This is extensively treated in a rigorous fashion in the book.

Let us return to quantum communication theory, which we mentioned early in this review. Armed with what we have just learned, we can now gain a deeper understanding of this theory. In a quantum communcation system one has a source T (say a laser beam), an information carrier or channel C (say a glass optical fiber), and a receiver R. In the absence of a signal, C is in a known state S (usually an equilibrium Gibbs state at a given temperature). Transmission of a signal is accomplished through an influence of T on C which forces a definite change in state for C. Usually there are parameters θ of T which can be varied (such as frequency or amplitude), and the resulting state S_θ depends on θ. In classical communication theory (such as radio waves) the states S_θ are given by probability distributions P_θ on the phase space of C. This is the classical statistical model. However, at optical frequencies quantum fluctuations become significant, and a quantum statistical model is required. In this case the states S_θ become density operators in a Hilbert space H. The signal obtained by the receiver is given by a state $S_{\hat{\theta}}$ which is a quantum distortion of the state S_θ. Thus, the receiver obtains an estimate $\hat{\theta}$ of the actual value θ of the transmitted signal. In this way the receiver performs a measurement of the parameter θ. Theoretical bounds for the accuracy of such measurements are of principal importance in optimal signal reception. We have previously indicated examples of such bounds.

In summary, this book presents a balanced blend of theory and applications. It moves smoothly from the basic concepts of quantum mechanics and operators on Hilbert spaces to the abstract theory of quantum estimation and to important recent practical applications. Each chapter ends with an interesting comments section which gives historical background and references. The book is motivated and readable. I recommend it.

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