$L^2$ HARMONIC FORMS AND A CONJECTURE OF DODZIUK-SINGER

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Let $M^n$ be a complete simply connected Riemannian manifold of sectional curvature $K_M$ satisfying $-a^2 \leq K_M \leq -1$, $a \geq 1$. Let $\mathcal{H}_p^2(M^n)$ denote the space of $L^2$ harmonic $p$-forms on $M$, i.e., $p$-forms $\omega \in \Lambda^p(M^n)$ such that

$$\Delta \omega = 0, \quad \int_{M^n} \omega \wedge * \omega = \int_{M^n} |\omega|^2 dV < \infty.$$ 

It is clear that $\mathcal{H}_p^2(M^n)$ is naturally isomorphic to $\mathcal{H}_2^{n-p}$ under the Hodge $*$ operator, and $\mathcal{H}_2^2(M^n) = 0$. Further, it is known [2] that $\mathcal{H}_2^2(M^n)$ naturally injects into the $L^2$-cohomology of $M^n$. Dodziuk and Singer (see [3, 4 and 6]) have conjectured that $\mathcal{H}_2^p(M^n) = 0$ if $p \neq n/2$ and $\dim \mathcal{H}_2^{n/2} = \infty$ if $n$ is even. An affirmative solution of this conjecture implies, by means of the $L^2$ index theorem for regular covers of Atiyah [1], a positive solution of the well-known Hopf Conjecture: If $M^{2m}$ is a compact manifold of negative sectional curvature, then $(-1)^m \chi(M^n) > 0$.

Dodziuk [3] has proved the $L^2$ form conjecture for rotationally symmetric metrics—in particular for the space forms $H^n(-a^2)$ of curvature $-a^2$. Donnelly and Xavier [5] have recently obtained results in case the curvature of $M^n$ is sufficiently pinched: They show $\mathcal{H}_2^p(M^n) = 0$ if $0 < p < (n - 1)/2$ and $a < (n - 1)/2p$.

In this note, we outline the construction of counterexamples to the $L^2$ form conjecture, in every dimension and degree except the middle. Our main result is

THEOREM. For any $n \geq 2$, $0 < p < n$ and $a > |n - 2p|$, with $a \geq 1$, there exist complete simply connected Riemannian manifolds $M^n$ with

$$-a^2 \leq K_M \leq -1$$ 

such that $\dim \mathcal{H}_2^p(M^n) = \infty$.

These manifolds have large isometry groups, $I(M) = O(2p - 1, 1) \times O(n - 2p + 1)$: the principal orbits have codimension $n - 2p$. However, $I(M)$ does not have discrete cocompact subgroups and thus $M^n$ cannot be used to construct counterexamples to the Hopf conjecture. There are quotients of the topological form $\overline{M}^{2p-1} \times \mathbb{R}^{n-2p+1}$, where $\overline{M}^{2p-1}$ is a compact manifold of curvature $-1$. 

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OUTLINE OF CONSTRUCTION. We define the manifolds $M^n$ to be warped products

$$M^n = H^{2p}(-a^2) \times_f S^{n-2p}(1),$$

where $S^{n-2p}(1)$ is the space form of curvature +1 and $f: H^{2p}(-a^2) \to R$, $f(x) = \sinh s(x)$, where $s$ is the distance to a fixed totally geodesic hyperplane $H^{2p-1} \subset H^{2p}(-a^2)$. The metric on $M^n$ is given by

$$ds^2 = ds_{H^{2p}(-a^2)}^2 + f^2 ds_{S^{n-2p}(1)}^2.$$

One easily verifies that $(M^n, ds^2)$ is a complete Riemannian manifold, diffeomorphic to $R^n$.

(i) CURVATURE OF $M$: Let $\{X_i\}$ be a local orthonormal framing of $H^{2p}(-a^2)$ by eigenvectors of $D^2 f$ and $\{V_j\}$ a local orthonormal framing of $S^{n-2p}(1)$. One may show that the family of 2-forms $\{X_i \land X_j\}$, $\{X_i \land V_j\}$, $\{V_i \land V_j\}$ diagonalizes the curvature operator $\mathcal{R}: \Lambda^2(TM) \to \Lambda^2(TM)$ with corresponding sectional curvatures $-a^2$, $-a \coth s \cdot \tanh as$, $-1$. In particular, the sectional curvatures of $M$ lie in the range $[-a^2, -1]$.

(ii) HARMONIC FORMS ON $M$: Let $\omega \in \Lambda^p(H^{2p}(-a^2))$ be invariant under reflection through $H^{2p-1}$ and extend $\omega$ to $M$ by defining it to be invariant under the isometric $SO(n-2p+1)$ action on $M$. One computes that

$$\Delta_M \omega = \Delta_{H^{2p}} \omega + (-1)^p [d \circ \iota_F - \iota_F \circ d] \omega$$

where $F = (n-2p) df/f$ is the negative of the mean curvature of $S^{n-2p} \subset M^n$ and $\iota$ denotes interior multiplication. We outline a procedure reducing the case of general $p$ to $p = 1$. First, note the identity

$$H^{2p}(-a^2) = H^2(-a^2) \times_g H^{2p-2}(-a^2),$$

where $g: H^2(-a^2) \to R$, $g(x) = \cosh ar(x)$, $r$ is the distance function to a fixed point $0 \in H^2(-a^2)$. Further, under this decomposition, $F$ is tangent to the $H^2(-a^2)$ factors. Set

$$\omega = \phi \land \eta, \quad \phi \in \Lambda^1(H^2(-a^2)), \quad \eta \in \Lambda^{p-1}(H^{2p-2}(-a^2)).$$

If $\eta$ is any harmonic $(p-1)$-form on $H^{2p-2}(-a^2)$, then $\omega$ satisfies (1) if and only if

$$\Delta \phi - [d \circ \iota_F - \iota_F \circ d] \phi = 0 \quad \text{on } \Lambda^1(H^2(-a^2)).$$

To study the solutions of (2), set $\phi = du$ and use the conformal equivalence of $H^2(-a^2)$ with $\Omega = \{(x, \theta): x \in R, \ \theta \in (-\pi/2, \pi/2)\}$ to obtain the equivalent equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\phi(\theta)}{\partial \theta^2} + \phi(\theta) \frac{\partial u}{\partial \theta} = 0,$$

where $\phi(\theta) = (1/f_1)(\partial f_1/\partial \theta)$ and $f_1 = f|_{H^2(-a^2)}$: explicitly,

$$f_1 = f_1(\theta) = \frac{1}{2} \left[ \frac{\alpha^{1/\alpha} - \beta^{1/\alpha}}{\cos^{1/\alpha} \theta} \right]$$

where $\alpha = 1 + \sin \theta$, $\beta = 1 - \sin \theta$. Note that $\phi$ degenerates on $\partial \Omega$. We may assume, without loss of generality, that $(n-2p) > 0$, so $\phi > 0$. 

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It is now quite straightforward to verify that (3) has solutions, smooth up to \( \partial \Omega \). If we conformally identify \( H^2(-a^2) \) with \( B^2(1) \) with the flat metric, one may produce an infinite-dimensional space of solutions of (3) with \( |du|_\infty < 1 \).

(iii) \( L^2 \) Estimate: First, we recall that \( |\omega|^2 = \int \omega \wedge * \omega \) is a conformal invariant for forms in the middle dimension. For \( \omega \) as above, we have

\[
\int_{M^n} |\omega|^2 = \int_{H^{2p} \times S^{n-2p}} |\omega|^2 f^{n-2p} dV_H dV_S
\]

\[
= \text{vol}(S) \int_{H^2 \times H^{2p-2}} |\phi|^2 |\eta|^2 f^{n-2p} dV_H^2 dV_{H^{2p-2}}
\]

\[
\leq \text{vol} S^{n-2p} \cdot \text{vol} B^{2p-2}(1) \cdot \int_{B^2} f^{n-2p} dV_B,
\]

where we have used the conformal equivalence of \( H^k(-a^2) \) with \( B^k(1) \), \( k = 2, 2p - 2 \) and assumed that \( \eta \) is a harmonic \((p - 1)\)-form with \( |\eta|_\infty \leq 1 \) with respect to the flat metric on \( B^{2p-2}(1) \), e.g. \( \eta = (1/(p-1)! \) \( dx_1 \wedge \cdots \wedge dx_{p-1} \).

One checks that

\[
\int_{B^2(1)} f^{n-2p} dV < c \cdot \int_0^{\pi/2} \cos^{-(n-2p)/a} \theta \, d\theta,
\]

so that if \((n-2p)/a < 1\), one has \( \int_{M^n} |\omega|^2 < \infty \).

Further discussion and examples will appear elsewhere.

REFERENCES


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