GENERALIZATIONS OF THE NEUMANN SYSTEM

BY R. J. SCHILLING

0. Introduction. The following observation, due to E. Trubowitz [7], illustrates an intimate relationship between spectral theory and Hamiltonian mechanics in the presence of constraints. Let \( q(s) \) be a real periodic function such that Hill’s operator,

\[
L = \left( \frac{d}{ds} \right)^2 - q(s),
\]

has only a finite number \( g \) of simple eigenvalues. There exist \( g + 1 \) periodic eigenfunctions \( x_0, \ldots, x_g \) and corresponding eigenvalues \( a_0, \ldots, a_g \) of \( L \) such that

\[
1 = \sum_{r=0}^{g} x_r^2 \quad \text{and} \quad q = -\sum_{r=0}^{g} (a_r x_r^2 + y_r^2),
\]

where \( y_r = dx_r/ds \). The equations \( Lx_r = a_rx_r \ (r = 0, \ldots, g) \) are equivalent to the classical Neumann system [7].

H. Flaschka [3] obtained similar results from a different point of view. His approach is based on the articles [2 and 5] of I. V. Cherednik and I. M. Krichever. The familiar Lax pairs, the constants of motion and the quadrics of the Neumann system emerge as consequences of the Riemann-Roch Theorem.

The purpose of our work is to apply Flaschka’s techniques to operators of order \( n \geq 2 \). We will be defining higher Neumann systems whose theory is closely tied to the spectral theory of linear differential operators of order \( n \). C. Tomei [9], using scattering theory, obtained some of our \( n = 3 \) formulas.

Preliminaries.

(1.1) RIEMANN SURFACE. Let \( R \) be a Riemann surface of genus \( g_R \) with a point \( \infty \) and a rational function whose divisor of poles \( (\lambda)_\infty \) is \( n^\infty \). We set \( \kappa = \lambda^{1/n} \). Then \( \kappa^{-1} \) is a local parameter vanishing at \( \infty \). Let \( W \) be the set of Weierstrass gap numbers of \( \infty \).

(1.2) ALGEBRAIC CURVES. We assume that \( R \) admits a second rational function \( z \) with the following 3 properties. There exists an integer \( N \geq 0 \) and an integer \( l \in \{1, 2, \ldots, n - 1\} \) relatively prime to \( n \) such that

\[
z = \lambda^{-N} \kappa^{-l}(z_0 + z_1 \kappa^{-1} + \cdots), \quad z_0 = 1, \text{ at } \infty.
\]

Let \( (z)_\infty = (0) + \cdots + (m), \ (r) \in R, \) be the divisor of poles of \( z \). Let \( a_r = \lambda(r) \).

We assume that each \( (r) \) is a simple pole and \( a_r \neq a_s \) whenever \( s \neq r \). We...
assume that the genus $g_R$ is related to $m, n$ and $l$ by the following important formula, $g_R = \frac{1}{2}(n-1)(2(m+1) - nN - (l+1))$. It is known that two rational functions on a Riemann surface satisfy a polynomial equation. Since that equation, it turns out, follows from the Baker function theory below, we need not discuss the existence of Riemann surfaces with the properties above.

Since $n$ and $l$ are relatively prime, there exist $r_j, s_j \in \mathbb{Z}$ such that $\lambda^r_j z^{s_j}$ has a pole of order $j$ at $\infty$. Let $t = (t_j | j \in W)$ be a vector of $g_R$ complex “time” parameters. Let $\theta = \sum_{j \in W} t_j^r_j z^{s_j}$.

(1.3) BAKER FUNCTIONS. Let $\delta$ be a positive nonspecial divisor of degree $g_R$ that does not meet $\infty$ and satisfies $L(\delta - \infty) = \{0\}$. It is known that there exists a unique function $\psi = \psi_\delta(t, p)$, called the Baker function of $\delta$, with the following two properties. $\psi$ is meromorphic in $R - \infty$ and any pole of $\psi$ lies in $\delta$. Near $\infty$, $\psi$ is given by $\psi e^{-\theta} = 1 + \xi_1(t)\kappa^{-1} + \xi_2(t)\kappa^{-2} + \cdots$, where the $\xi_j$ are functions analytic on an open subset of $C^R$ containing $t = 0$.

(1.4) DUAL BAKER FUNCTION. By the Riemann-Roch Theorem there exists a unique abelian differential $\Omega$ and a positive nonspecial divisor $\delta'$ of degree $g_R$ such that $(\Omega) = \delta + \delta' - 2\infty$ and $\Omega = \kappa^2(1 + O(\kappa^{-2})) d\kappa^{-1}$ at $\infty$. Let $\phi = \psi_{\delta'}(-t, p)$. We will refer to $\phi$ as the Baker function dual to $\psi$ and $\delta'$ will be called the dual divisor [2].

(1.5) NEUMANN SYSTEMS. There exists a linear differential operator $L$ of order $n$ in $d/dt_1$ and, for each $j \in W$, a linear differential operator $\bar{L}_j$ of order $j$ in $d/dt_j$ such that

\begin{equation}
L(t)\psi(t, p) = \lambda(p)\psi(t, p) \quad \text{and} \quad \bar{L}_j(t)\psi(t, p) = \frac{\partial \psi}{\partial t_j}(t, p).
\end{equation}

Let $L^*$ be the formal real adjoint of $L$ (for instance, $(qD^j)^* = (-1)^jD^j q$).

The article [2] contains a clever proof of the following formulas:

\begin{equation}
L(t)^*\phi(t, p) = \lambda(p)\phi(t, p) \quad \text{and} \quad \bar{L}_j(t)^*\phi(t, p) = \frac{\partial \phi}{\partial t_j}(t, p).
\end{equation}

We are now in position to define the main object of our analysis. Let $\rho_r = \text{Res}_{(r)}(2\Omega)$ and choose constants $\alpha_r, \beta_r \in C^*$ such that $\rho_r = \alpha_r\beta_r$. We evaluate the Baker functions $\psi$ and $\phi$ over the poles of $z$ to make the following definitions:

\begin{equation}
x_r^t(t) = \alpha_r\psi(t, r) \quad \text{and} \quad u_r^t(t) = \beta_r\phi(t, r), \quad r = 0, \ldots, m.
\end{equation}

Let $m \in C^{2n(m+1)}$ be the point whose coordinates are $x_r^t, u_r^t$ and their first $n - 1$ derivatives with respect to $t_1$. We are concerned with the equations obtained from (1.5.1) by setting $p = (r), r = 0, \ldots, m$.

(1.6) SOLITON EQUATIONS. The integrability condition of the simultaneous linear equations (1.5.1) is the partial differential equation

\begin{equation}
\frac{\partial L}{\partial t_j} = [\bar{L}_j, L], \quad j \in W.
\end{equation}

The Lax equation usually suggests that certain spectral data associated to $L$ are preserved in time. In the present setup it is the Riemann surface $R$ that is preserved. Two of the equations (1.6) are important in their applications to soliton mathematics. If $n = 2$ and $j = 3$, (1.6) is the Korteweg-de Vries
Results.

(2.1) SYMPLECTIC MANIFOLD AND TRACE FORMULAS. The differential \( \eta = \psi_j^{(s)}(\phi_j^{(s)})^\Omega \) is meromorphic because the exponents of \( \psi \) and \( \phi \) at \( \infty \) cancel. The meromorphic differential \( \eta = \lambda^k z \eta \) has simple poles in \( (z)_\infty \) and it may have a pole at \( \infty \). Let \( C_\eta = \sum_{p \in R} \text{Res}_p(\eta) \). The classical formula \( \sum_{p \in R} \text{Res}_p(\eta) = -\text{Res}_\infty(\eta) \) expresses \( \text{Res}_\infty(\eta) \) in terms of \( m \). If \( \text{Res}_\infty(\eta) \) is constant (in \( t \)) the equation \( C_\eta = 0 \) defines a hypersurface in \( C^{2n(m+1)} \).

The functions \( C_\eta \) with \( \text{Res}_\infty(\eta) \) constant are called constraints.

(2.1.1) THEOREM. The algebraic subset \( M \) of \( C^{2n(m+1)} \) defined in terms of the quadratic constraints \( C_\eta = 0 \) is a symplectic manifold. The dimension of \( M \) is given by \( \dim(M) = 2g_R + 2(m+1) \).

(2.1.2) THEOREM. The coefficients of \( L \) are expressible in terms of the point \( m \) associated with the Baker function and the poles of \( z \).

It follows then that the equations (1.5.1) with \( p = (r), \) \( r = 0, \ldots, m, \) define \( g_R \) autonomous vector fields \( X_j^r, j \in W, \) on \( M \). The \( (n = 2) \) vector field \( X_1^r \) is a generalization of the Neumann system [3 and 4].

(2.2) LAX EQUATIONS. One of the nicest results of Flaschka’s work is a systematic derivation of the well-known Neumann-Lax pairs. The best explanation for the existence of the Neumann-Lax pairs comes from Krichever’s theory of commutative rings of matrix differential operators. The divisor \( \Delta' = \delta' + (z)_0 - \infty \) is nonspecial and its degree is \( g_R + m \). Following [4] we call \( \Delta' \) the augmented dual divisor. According to [8], there exists a vector function \( \Phi = (\Phi^0, \ldots, \Phi^m)^T \) with the following two properties. \( \Phi \) is meromorphic in \( R - (z)_\infty \) and any pole in \( \Phi \) lies in \( \Delta' \). Near \( (r) \), \( \Phi^s \) is given by \( \Phi^s e^{-\theta} = \alpha_r \delta_{r,s} + O(z^{-1}) \). Let \( (; ;) \) be the bilinear form associated to \( L \) by the Lagrange identity, \( d(\cdot; g)/dt = L\cdot g - g \cdot L^* g \). H. Flaschka discovered the \( n = 2 \) version of the very beautiful formula,

\[
\Phi^r(t, p) = (x^r_1(t); \phi(t, p)) \frac{z^{-1}(p)}{\lambda(p) - \alpha_r} e^{\theta(t, p)}.
\]

According to Krichever there exists an \( (m+1) \times (m+1) \) matrix \( B_j \) that depends polynomially on \( z \) such that \( \Phi_{t_1} = B_j \Phi \). Using Flaschka’s formula (2.2.1) we are able to express \( B_j \) in terms of \( m \). The function \( \lambda z^n \) belongs to the ring \( H^0(R - (z)_\infty, O_R) \). Thus according to Krichever there exists an \( (m+1) \times (m+1) \) matrix \( L \) that depends polynomially on \( z \) such that \( L \Phi = \lambda z^n \Phi \). The Lax equation \( \mathbf{L}_{t_j} = [B_j, L] \) is immediate. Our explicit formulas show that \( L \) is a rank \( n \) perturbation of the diagonal matrix \( az^n \) in that the range of \( L - az^n \) is spanned by \( x_1, \ldots, x_n \). The \( (n = 2) \) and \( B_1 \) generalize the Neumann-Lax pairs in [1, 3 and 4].

We have \( \Delta' - (\phi)_\infty \geq 0 \) and therefore \( \phi^0 e^{\theta} \) belongs to the linear space of Baker functions spanned by the components of \( \Phi \). This observation led
Flaschka to the $n = 2$ version of the following formula:

$$\phi(t,p)e^\theta = \sum_{r=0}^{m} u_n^r(t)\Phi^r(t,p) = \langle u_n(t),\Phi(t,p) \rangle.$$  

(2.2.2)

The formula has two applications. We use (2.2.2) to obtain explicit formulas for the operators $\tilde{L}_j$. Such formulas were one of Cherednik’s objectives [2]. When $\Phi$ is eliminated from (2.2.2) by use of (2.2.1) we obtain the following result.

(2.2.5) **THEOREM.** There exists an $n \times n$ matrix $Z = Z(m, \lambda)$, rational in $\lambda$, whose spectrum is independent of $t$. The algebraic relationship (1.2) between $\lambda$ and $z$ is given by the characterization polynomial $\det(Z - zI) = 0$.

(2.3) **COMPLETE INTEGRABILITY.** The $m + 1$ Hamiltonians $(x_1^r; u_n^r)$, $r = 0, \ldots, m$, are rather trivial involutive constants of motion. A reduction of $M$ by these Hamiltonians defines a symplectic manifold which, by (2.1.1), has dimension $2g_R$. We use the fact that the eigenvalues of $L$ and $Z$ are constants of the motion to construct a Hamiltonian $\tilde{H}_j^*$ for each vector field $X_j^*$, $j \in W$.

(2.3.1) **THEOREM.** The $g_R$ Neumann vector fields $X_j^*$ of (2.1.4) form a completely integrable Hamiltonian system.

It is known that the level surface $M_C \defeq \{m^* \in M | H_j(m) = c_j \}$ of a completely integrable system, if real and compact, is a torus. Our last result is concerned with the structure of these energy level sets.

(2.3.2) **THEOREM.** The level surface of the reduced manifold is locally isomorphic to the Zariski-open subset, Jacobian-(theta divisor) of the Jacobian variety of the algebraic curve given by $\det(Z(A) - zI)$.

The idea in the proofs of (2.3.1,2) is an algebro-geometrical version of the solitonic inverse scattering transform. Let $M$ be one of the symplectic manifolds of Theorem (2.1.1). We assign to each point $m \in M$ an algebraic curve $C$ and a divisor $\delta = \delta_m$ on $C$. The isomorphism of Theorem (2.3.2), called the divisor map, is given by

$$m \in M \rightarrow (C, \delta) \rightarrow (\text{Jac}(C), A(\delta))$$

where $A$ is the Abel map. It contains a method for linearizing the equations of motion. The important ideas can be found in [1 and 5]. We apply McKean’s pole conditions [6, p. 624] to make certain results, especially the description of $\delta$, more explicit.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803