BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 14, Number 2, April 1986

REALIZING SYMMETRIES OF A SUBSHIFT OF FINITE TYPE BY HOMEOMORPHISMS OF SPHERES

BY J. B. WAGONER[1]

Let A be a finite, irreducible, zero-one matrix and let $\sigma_A: X_A \to X_A$ be the corresponding subshift of finite type [**F**]. Recall from [**F**] that a Smale diffeomorphism is one with a hyperbolic zero-dimensional chain recurrent set. A well-known theorem of Williams-Smale [**Wi**] says that there is a Smale diffeomorphism $F_A: S^3 \to S^3$ so that σ_A is topologically conjugate to the restriction of F_A to the basic set of index one occurring as part of the spectral decomposition. Let $\mathrm{Aut}(\sigma_A)$ denote the group of symmetries of σ_A—that is, the group of homeomorphisms of X_A which commute with σ_A. Here is the corresponding global realization result for these symmetries.

THEOREM. *Assume* $4 < q$ *and let* $1 < e < q - 2$. *Then there is a Smale diffeomorphism* $F_A: S^q \to S^q$ *with a basic set* Ω_e *of index* e *(along with other basic sets of index* 0, $e+1$, q*) together with a topological conjugacy between* σ_A *and* $F_A|\Omega_e$ *so that, given any symmetry* g *in* $\mathrm{Aut}(\sigma_A)$, *there is a homeomorphism* $G: S^q \to S^q$ *satisfying*
(A) G *commutes with* F_A *on all of* S^q,
(B) $G|\Omega_e = g$ *under the identification between* $\mathrm{Aut}(F_A|\Omega_e)$ *and* $\mathrm{Aut}(\sigma_A)$.

The motivation and the idea for the proof of this geometric result came by analogy from algebraic K-theory and pseudo-isotopy theory. The proof uses Williams' notion of strong shift equivalence [**W1, F**], the contractible simplicial complex P_A of topological Markov partitions for σ_A [**W1**], and structural stability for Smale diffeomorphisms [**R, Ro**]. We would like to thank C. Pugh for useful discussions about the stability theorem.

The group $\mathrm{Aut}(\sigma_A)$ is often rather large. For example, $\mathrm{Aut}(\sigma_2)$ for the Bernoulli 2-shift σ_2 has been known [**H**] for some time to contain every finite group and to have elements of infinite order not a power of σ_2. Recently, Boyle and Lind have shown it contains the free nonabelian group on infinitely many generators. Therefore, the group of *homeomorphisms* of S^q commuting with a certain F_2 is large when $4 < q$. Incidentally, at the present time not much is really known about the structure and other algebraic or homological properties of $\mathrm{Aut}(\sigma_2)$. For some information see [**BK**] or [**W1**]. An open and long-standing conjecture is that $\mathrm{Aut}(\sigma_2)$ is generated by σ_2 and elements of finite order.

Here is a rough idea of the proof of the Theorem. The details will appear in [**W2**]. Let P be an $m \times m$ zero-one matrix and let Q be an $n \times n$ zero-one matrix. Suppose there is an $m \times n$ zero-one matrix R and an $n \times m$ zero-one

Received by the editors September 30, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C35, 20B27.
[1]Partially supported by the NSF.

matrix S so that $P = RS$ and $Q = SR$. As in [**Wi**], this determines a specific conjugacy $c_R: (X_P, \sigma_P) \to (X_Q, \sigma_Q)$ sending $x = \{x_i\}$ in X_P to $c_R(x) = \{c_R(x)_i\}$ in X_Q, where $c_R(x)_i$ is the unique k such that $R(x_i, k)S(k, x_{i+1}) = A(x_i, x_{i+1}) = 1$. Similarly for c_S. In fact, $c_S c_R = \sigma_P$ and $c_R c_S = \sigma_Q$, so that $c_R \sigma_P = \sigma_Q c_R$ and $c_S \sigma_Q = \sigma_P c_S$. We call c_R and c_S *elementary symbolic conjugacies.*

On the topological side, let $S^q(m)$ be the standard q-sphere equipped with a fixed handle decomposition with one handle of index zero, m handles of index e, m cancelling handles of index $e+1$, and one handle of index q. Similarly for $S^q(n)$. One then constructs a Smale diffeomorphism $C_R: S^q(m) \to S^q(n)$ which is fitted both on the handles of index e and the handles of index $e+1$ according to the geometric intersection matrix R. Again, similarly for C_S. This is done in such a way that the composition $D_P = C_S C_R: S^q(m) \to S^q(m)$ is also a Smale diffeomorphism fitted on the e-handles and $(e+1)$-handles according to the matrix $P = RS$ and $D_Q = C_R C_S: S^q(n) \to S^q(n)$ is fitted according to $Q = SR$. Observe that $C_R D_P = D_Q C_R$ and $D_P C_S = C_S D_Q$, and therefore C_R and C_S are smooth conjugacies between D_P and D_Q. We call these *elementary smooth conjugacies.*

Now consider a Smale diffeomorphism $F_P: S^q(m) \to S^q(m)$ which is fitted on the e-handles and $(e+1)$-handles by the matrix P. In general, of course, $F_P \neq D_P$. However, under the assumption that $1 < e < q-2$ we are able to carefully construct F_P, C_R, and C_S in such a way that there is a one-parameter family of Smale diffeomorphisms $F_P(t)$, each of which is fitted on the e-handles and $(e+1)$-handles by the matrix P, so that $F_P(0) = F_P$, $F_P(1)$ is equal to D_P on a neighborhood of the $(e+1)$-skeleton, and both $F_P(1)$ and D_P have the point at infinity as a source. Methods of stability theory [**R, Ro**] can then be used to produce a topological (*not* smooth) conjugacy between F_P and D_P. We call this a *stability conjugacy.* Similarly, there is a stability conjugacy between D_Q and F_Q, so that we then get a topological conjugacy F_P and F_Q.

The main theorem is proved by first showing that any symmetry g in $\mathrm{Aut}(\sigma_A)$ can be obtained as the composition of a chain of elementary symbolic conjugacies and powers of shifts, and then by showing this can be mirrored compatibly with a corresponding chain of elementary smooth conjugacies, stability conjugacies, and powers of certain intermediate F_P for different matrices P. The chain starts with the original F_A which is fixed and eventually comes back to it. The composition of the various conjugacies and powers of F_P in the chain give the required homeomorphism G.

The main theorem may well be valid on S^4 also, but our argument seems to require $4 < q$.

REFERENCES

[**BK**] M. Boyle and W. Krieger, *Periodic points and automorphisms of the shift*, preprint, Univ. of Maryland/Inst. Angew. Math. der Univ. Heidelberg, 1985.

[**F**] J. Franks, *Homology and dynamical systems*, CBMS Regional Conf. Ser. in Math., no. 49, Amer. Math. Soc., Providence, R.I., 1982.

[**H**] G. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3** (1969), 320–375.

[**R**] J. Robbin, *A structural stability theory*, Ann. of Math. (2) **94** (1971), 447–493.

[**Ro**] C. Robinson, *Structural stability for C^1 diffeomorphisms*, J. Differential Equations **22** (1976), 28–73.

[**W1**] J. B. Wagoner, *Markov partitions and K_2*, preprint, Univ. of California, Berkeley, 1985.

[**W2**] ____, *Realizing symmetries of a shift*, preprint, Univ. of California, Berkeley, 1985.

[**Wi**] R. F. Williams, *Classification of subshifts of finite type*, Ann. of Math. (2) **98** (1973), 120–153; errata **99** (1974), 380–381.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720