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Gauss, a biographical study, by W. K. Bühler, Springer-Verlag, Berlin, Heidelberg, New York, 1981, viii + 208 pp., $19.80. ISBN 0-387-10662-6

Most contemporary mathematicians would demur if asked to name the greatest mathematician of the last fifty years. However, if the period were lengthened to a hundred years, most mathematicians would become less reticent and many would mention the name of Hilbert. If the period were lengthened to two hundred years, reticence would largely disappear and almost all mathematicians would immediately give the name of Carl Friedrich Gauss (1777–1855). In addition to his top ranking in mathematics, Gauss also ranks very high in such other scientific fields as astronomy, computing science, geodesy, physics, and statistics.

In a very real sense Gauss marks the beginning of the present epoch in mathematics. Gauss was the first person to give a proof of the Fundamental Theorem of Algebra which present-day mathematicians would find acceptable and the first person to write down the unique factorization theorem for ordinary integers in the form which we use today. Gauss aimed for a standard of mathematical precision comparable to that generally accepted today, in contrast to the occasional sleight-of-hand practiced by many of his predecessors. But, aside from this modern-seeming insistence on perfection of form, his writings have an almost contemporary flavor simply because his work has had a tremendous influence in shaping the subsequent trends of mathematical research. A symposium on the Mathematical Heritage of Carl Friedrich Gauss would be an impossibility, since it would have to cover a large part of present-day mathematics, including such widely separated fields as number theory, field theory and polynomials, linear algebra, functions of a complex variable, potential theory, special functions, calculus of variations, foundations of geometry, differential geometry, probability theory, numerical analysis, and dynamical systems.

In addition to the extraordinary level of his basic mathematical talent, several other ingredients went into the making of Gauss's outstanding mathematical career. First, he received solid financial support for his study and research between the ages of 15 and 30. Second, he was largely unaffected by the political upheavals of his time, specifically the Napoleonic wars. Third, he enjoyed good health, reasonable longevity, and a relatively stable domestic life. Finally, he was appointed to a secure position as director of the Göttingen astronomical observatory at age 30 and held that post for the rest of his life.

Gauss's attendance at secondary school in his home town of Brunswick (1792–1795), his studies at the University of Göttingen in the neighboring kingdom of Hanover (1795–1798), and his nine-year period of postdoctoral research in Brunswick (1798–1807) were all supported by grants from Carl Wilhelm Ferdinand, Duke of Brunswick. Gauss's ability to spend essentially all of his twenties in postdoctoral research, free of financial worries and academic duties, was decisive in forming a basis for his subsequent academic career. Gauss's great number-theoretic book, *Disquisitiones Arithmeticae*, published in 1801, was conceived partially as a scientific report to the Duke. Surely it must rank as the number-one grant report of all time. (There appears to be no evidence that the Duke asked for time-and-effort reporting.) The Duke died in 1806 from injuries suffered during his defeat by Napoleon in the battle of Auerstädt. Fortunately for Gauss, negotiations for his appointment at Göttingen were already under way when the Duke died. The fact that, during these negotiations, Göttingen moved from the rule of the notorious Hanoverian king George III to occupancy by the Prussian forces and then to French domination (via the kingdom of Westphalia under Jerome Napoleon) does not seem to have upset the negotiations in any essential way.

Gauss made his mark as a top-level mathematician very early in life and then was able to retain his high standing for almost sixty years. His first substantial mathematical conquests came in 1796, the year of his 19th birthday. First, he established the constructibility of the regular 17-gon with ruler and compass by deriving the specific formula

$$\begin{aligned}16\cos\frac{2\pi}{17} &= -1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\\ &\quad +2\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}\,.\end{aligned}$$

Second, he obtained the result noted in his diary as

$$\text{EYPHKA!}\quad \text{num} = \Delta + \Delta + \Delta,$$

i.e., every positive integer is expressible as a sum of three triangular numbers. (A triangular number is half the product of two consecutive integers.) Third, he obtained the first correct proof of the Quadratic Reciprocity Law, which had been conjectured earlier by Euler and Legendre. Thus 1796 was a vintage year by any standard. In 1799 Gauss received a Ph.D. from the University of Helmstedt (in the duchy of Brunswick) for a thesis giving his first proof of the Fundamental Theorem of Algebra. In 1800 he completed work on *Disquisitiones Arithmeticae*, the most important single book in the history of number theory; it contained, for example, proofs of the three results from 1796 quoted above, not to mention Gauss's arithmetical theory of binary quadratic forms.

Finally, in 1801 he achieved global fame in the world of science by determining the orbit of the planetoid Ceres from a small number of observations made by the Italian astronomer Piazzi before it disappeared behind the sun; this enabled Gauss to give a sensationally correct prediction as to where in the sky Ceres would subsequently reappear. This event has been applauded by most but decried by some (e.g., E. T. Bell) as marking Gauss's first departure from the ivory tower of pure mathematics and his first success in applying mathematical methods in other sciences. Whether this step was a tragedy or not, Gauss certainly did turn more and more to such activities as observational astronomy, geodetic surveying, and magnetic mapping as he grew older and settled into his position as director of the Göttingen astronomical observatory.

The primary literature related to Gauss's career is extremely vast. The twelve volumes of his collected works [**6**] contain all of his published work (mostly in Latin) and apparently all of the unpublished material left behind at his death which has any scientific value. Many volumes of correspondence have been published, including correspondence with Bessel, Wolfgang Bolyai, Encke, Gerling, Sophie Germain, Nicolai, Olbers, Schumacher, and Alexander von Humboldt. The *Disquisitiones Arithmeticae* (*Werke*, I) has also been published in French, German, and English translations; Maser's German translation [**7**] is the most valuable, since it also contains German translations of Gauss's published papers in number theory and most of his unpublished manuscripts in number theory as well. The series *Ostwald's Klassiker der exakten Wissenschaften* includes over half a dozen mini-volumes devoted to various crucial parts of Gauss's work, e.g., [**8** and **9**]. Thus, the primary record of Gauss's career is as good as could reasonably be hoped for.

In addition there is an extensive secondary literature. The most significant single work is probably Klein's lengthy chapter on Gauss in [**11**]. Following Klein's initiative, a series of articles on various aspects of Gauss's work were written during the latter part of the nineteenth century and the early part of the twentieth century by Bachmann, Bolza, Brendel, Fraenkel, Galle, Geppert, Maennchen, Ostrowski, Schaefer, Schlesinger, and Stäckel. These essays were published in Volumes X and XI of the *Werke* and most of them initially appeared in the *Göttinger Nachrichten*. Finally, two volumes [**13**, **16**] were published in connection with the hundredth anniversary of Gauss's death in 1955, one in the USSR and one in Germany. These contained articles on various areas of Gauss's work by Blaschke, Delone, Falkenhagen, Gnedenko, Kähler, Klingenberg, Kochendörffer, Markushevitch, Norden, Rieger, Salié, Schröder, Subbotin, and Volk.

Actual biographies of Gauss are few and far between. The only account by a contemporary of Gauss is the booklet [**15**], published immediately after Gauss's death by the geologist Sartorius, who did not have much understanding of Gauss's mathematical work. The *Werke* do not contain a biography, presumably because the editors did not find Sartorius' biography adequate. In English the only biographies appear to be Bell's chapter on Gauss in [**2**], Hall's small book [**10**], and Dunnington's lengthy book [**5**].

Many are familiar with Bell's biography, since it has been reprinted in Volume 1 of [**12**]. It displays Bell's breezy style at its best (and also his idiosyncratic point of view). In view of the extensive documentation which exists in the case of Gauss, the reader is spared the speculative passages which mar some of Bell's other biographies. In fact, Bell's chapter on Gauss seems remarkably accurate.

Hall's book might better be called "Gauss for the Million." It is an unabashed attempt at popularization and in this effort is remarkably successful. It assumes only a knowledge of basic undergraduate mathematics, but attempts to give the reader some real insight into Gauss's scientific accomplishments. It makes effortless reading for a professional mathematician and is the book on Gauss for a mathematician to read if he or she does not wish to devote much time to the enterprise.

Dunnington's book is a very detailed personal biography of Gauss with an old-fashioned charm, but is very sketchy on the mathematical side. Dunnington's training was in German literature and he was not well equipped to go deeply into scientific matters. For example, he refers to the Quadratic Reciprocity Law as "Legendre's Theorem." (Legendre did attempt proofs of the Quadratic Reciprocity Law, but his proofs contained serious gaps.) Dunnington probably gives more details about Gauss's life than most mathematicians would care to know. For example, he gives full genealogical tables, a list of all the books Gauss took out of the Göttingen library, and a list of all the courses which Gauss gave at Göttingen. In Dunnington's book the reader will discover that Gauss attended concerts in Göttingen by Weber, Paganini, Liszt, and Jenny Lind, but apparently never met Brahms while the latter was a student at Göttingen. Dunnington worked on this biography off and on for thirty years prior to its publication in 1955, including a year spent in Gauss's living quarters in the Göttingen astronomical observatory. (Interestingly enough, Dunnington's Ph.D. thesis **[4]** on the relationship of Jean Paul to Karl Philipp Moritz was not totally unrelated to Gauss, since Jean Paul was one of Gauss's favorite writers and Moritz wrote the novel *Anton Reiser*, which, according to Bühler, is "an excellent source for the social and cultural background of Gauss's youth.")

Of the book under review Bühler writes in his introduction "This biography is addressed to the contemporary mathematician and scientist, not to the historian of science or the psychologist collecting the scalps of great men. Its aim is modest—this is not an attempt to write a definitive 'Life of Gauss.' At the same time, it is immodest if not grandiose to try to select from Gauss's life and work those facets which are of contemporary interest and palatable to a reader who is not primarily historically motivated nor has any immediate reasons why he should be." It is fortunate that Bühler's aims were so reasonable, since otherwise the book could never have been written. A full-scale scientific biography of Gauss would probably require twenty years of concentrated effort by a scientist of Fields Medal caliber.

The result of Bühler's efforts is, as might have been predicted, somewhat uneven but is definitely worthwhile and interesting reading for any serious mathematician. The 25 chapters vary considerably in ease of reading. Many chapters read very well, but some of the chapters which go into mathematical detail are rather tough going. One has the impression that the various chapters were written at different times, and, as a result, they do not always hang together very well. The author does not have the writing skill of E. T. Bell or Constance Reid, say, but his best passages are very fine.

The strengths and weaknesses of Bühler's book can be illustrated by discussing the treatment of Gauss's number-theoretic work, specifically, the *Disquisitiones Arithmeticae*, Gauss's numerous proofs of the Quadratic Reciprocity Law, and Gauss's conjecturing of the prime-number theorem.

Bühler devotes two chapters to summarizing the contents of the *Disquisitiones Arithmeticae*. The reviewer found these chapters to be mostly indigestible, not very informative, and not totally accurate. For example, the discussion of Gaussian sums on page 30 is marred by misprints and a failure to explain the notation, while the discussion of Gaussian sums on page 76 is marred by the omission of the crucial condition $p \equiv 1 \pmod 4$.

The two proofs of the Quadratic Reciprocity Law which are given in the *Disquisitiones Arithmeticae* are well described by Bühler (pp. 22, 23, 26, 33). However, the discussion on page 34 of the four subsequently published proofs (*Werke* II, pp. 1–8, 9–45, 51–54 and 55–59) appears muddled. For example, Bühler does not mention what is now called Gauss's lemma, which was the basis for Gauss's third and fifth proofs (and for practically all proofs given in number theory books and courses today). Also he does not make clear that the fourth proof came as a by-product of Gauss's studies of Gaussian sums (in the famous 1811 paper "Summatio quarundam serierum singularium," in which after many years of effort, he finally eliminated an ambiguity of sign in the formulas for Gaussian sums). A full discussion of the various proofs may be found in Chapter 6 of Bachmann's book [**1**].

In a letter to Encke dated December 24, 1849 Gauss asserted that he had conjectured the prime-number theorem on empirical grounds when he was 15 or 16 years of age. (The prime-number theorem asserts that if $\pi(x)$ is the number of primes not exceeding the positive number x, then

$$\lim_{x\to\infty}\left\{\pi(x)\Big/\int_2^x(\log t)^{-1}\,dt\right\}=1.$$

It was proved independently by Hadamard and de la Vallée-Poussin in 1896.) This assertion to Encke was substantiated by the discovery in the early years of this century of a handwritten entry dated May, 1796 by Gauss in the back of his copy of a book of logarithmic tables. In contrast to Hall and some other writers, Bühler gives little space to this privately-made conjecture of Gauss and, in the reviewer's opinion, rightly so. The prime-number conjecture was made publicly (in a slightly garbled form) by Legendre at about the same time and its surfacing then was more the result of the appearance of several tables of primes in the latter part of the eighteenth century than of special insight on the part of Gauss and Legendre. Apparently Gauss did not make any theoretical contact with the prime number theorem and this probably inhibited him from mentioning the conjecture publicly. It is easy both to applaud Legendre's rushing into print with the conjecture and also to understand Gauss's reticence to do so. In any case any question of priority in conjecturing the prime-number theorem became of secondary interest after Chebyshev [**3**] and Riemann [**14**] made substantial contact with the prime-number theorem in the 1850s by real-variable and complex-variable methods respectively. (Chebyshev attributed the conjecture to Legendre, but Riemann attributed it to Gauss.)

In his introduction Bühler writes, "We hope the global picture is satisfactory even if there are many inconsistencies, omissions, and mistakes in detail." The reviewer believes that the author's hope is fulfilled. This book will undoubtedly be a disappointment to the reader who feels that Gauss's exalted talents and accomplishments merit a superhuman biography written by a mythical amalgam of C. L. Siegel and C. P. Snow. For the reader who is willing to face reality and read an honest attempt by a mere mortal to present Gauss and his work to the late twentieth century mathematical community, this book can give much pleasure and enlightenment. The reviewer genuinely enjoyed reading it.

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