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On a new method of analysis and its applications, by Paul Turán, John Wiley & Sons, Inc., Somerset, New Jersey, 1984, xvi + 584 pp., $49.95. ISBN 0471-89255-6

This book has a remarkable history. The present edition was announced twenty-five years before its publication. The first version of Turán's book was published in 1953 in Hungarian and in German. An improved Chinese edition followed in 1956. The theory was developed so rapidly that only a few years later these editions were out of date. In 1959, in a footnote of an article [**8**], Turán announced a completely rewritten English edition. He wanted to cover all the theory and major applications of the power sum method which bears his name. A series of new inequalities and applications, found by Turán and other mathematicians, among whom Atkinson, de Bruijn, Cassels, Erdős, Gaier, Halász, Pintz, van der Poorten, Sós, Stark, and Uchiyama, led him to extend the manuscript and to rewrite old parts of the manuscript repeatedly. At his death on September 26, 1976 he left a carefully organized manuscript comprising 57 sections. Sections 1–37 (except for Section 26) were in final form. For the remaining sections he had indicated the intended contents. These sections were written by Halász and Pintz following Turán's intentions as far as possible. Mrs. Sós took care of the further coordination. These efforts resulted in the publication of the English edition, eight years after Turán's death. The present book has 584 pages compared to 196 pages of the German edition. It is a unified treatise of the theory to which Turán devoted a considerable part of his life.

The roots of Turán's method are the roots of prime number theory. The Prime Number Theorem states that the number $\pi(x)$ of primes at most x is asymptotically equal to

$$\operatorname{li} x := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

The theorem is equivalent to the nonvanishing of the Riemann zeta function $\zeta(s) = \zeta(\sigma + it)$ at the line $\sigma = 1$. The still unproved Riemann Hypothesis states that $\zeta(s) \neq 0$ for $\sigma > 1/2$. For $\sigma > 1$ we have $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ and $\zeta(s) \neq 0$ because of Euler's product formula for $\zeta(s)$. In 1911 H. Bohr [**2**] proved the surprising fact that

$$\inf_{\sigma > 1} |\zeta(\sigma + it)| = 0.$$

In the same year Bohr [**1**] solved the problem of Lindelöf whether or not $\zeta(s)$ is bounded for $\sigma > 1$, $|t| \geqslant 1$ by proving its unboundedness. Bohr's proofs are similar, but for the latter result Bohr used Dirichlet's theorem and for the former Kronecker's theorem from the theory of diophantine approximation. An essential difference between these theorems is that there is a localization in Dirichlet's, but not in Kronecker's theorem. Such a localization enables one to prove the stronger inequality

$$\overline{\lim_{\substack{t\to\infty\\ \sigma>1}}} \frac{|\zeta(\sigma+it)|}{\log\log t} > \frac{1}{10}.$$

The essence of the proof of this inequality is embodied in the result that for positive real numbers $b_1, \ldots, b_n$, and arbitrary real numbers $\theta_1, \ldots, \theta_n$, the following inequality holds

$$\max_{1\leqslant t\leqslant 5^n}\left|\sum_{j=1}^{n} b_j e^{i\theta_j t}\right| \geqslant \left(\cos\frac{2\pi}{5}\right)\sum_{j=1}^{n} b_j. \tag{1}$$

Just before and during the dark days of World War II, also in a period that he was detained by the Nazis, Turán contemplated results of type (1). This led to his First and Second Main Theorems. The First Main Theorem states that, for arbitrary nonnegative integer m and complex b_j and z_j with $\min_j|z_j| = 1$, the inequality

$$\max_{\nu=m+1,\ldots,m+n}\left|\sum_{j=1}^{n} b_j z_j^{\nu}\right| \geqslant \left(\frac{n}{2e(m+n)}\right)^{n}\left|\sum_{j=1}^{n} b_j\right| \tag{2}$$

holds. Note that the length of the range of ν is minimal for a nontrivial result. Around 1960 Makai and de Bruijn computed, independently of each other, the best coefficient of $|\sum_{j=1}^{n} b_j|$, thereby showing that the factor $2e$ cannot be replaced by a smaller number. Turán's Second Main Theorem concerns the normalization $\max_j|z_j| = 1$. Since the left-hand side of (2) may now vanish (take e.g., $z_1 = 1$, $z_2 = \cdots = z_n = 0$, $b_1 = 0$, $b_2 = \cdots = b_n = 1/(n-1)$), the formulation becomes more complicated. After a series of improvements of the constants, the Second Main Theorem reads as follows. For arbitrary nonnegative integer m and complex b_j and z_j with $1 = |z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_n|$, the inequality

$$\max_{\nu=m+1,\ldots,m+n}\left|\sum_{j=1}^{n} b_j z_j^{\nu}\right| \geqslant 2\left(\frac{n}{8e(m+n)}\right)^{n}\min_{j=1,\ldots,n}|b_1 + \cdots + b_j| \tag{3}$$

holds. The constant $8e$ need not be optimal, but cannot be replaced by a constant less than $4e$. There are numerous variations and extensions.

In view of applications Turán wanted lower bounds for $\mathrm{Re}(\sum_{j=1}^{n} b_j z_j^{\nu})$ too. The condition

$$\kappa \leqslant |\operatorname{arc} z_j| \leqslant \pi \text{ for } j = 1,\ldots,n \text{ with } 0 < \kappa \leqslant \pi/2 \tag{4}$$

appeared to be sufficient to obtain nontrivial results. Again there are two main theorems, the so-called One-Sided Theorems, one with normalization $\min_j|z_j| = 1$ and one with $\max_j|z_j| = 1$. We state a simple form of the latter, Turán's Third Main Theorem. Let m be a nonnegative integer, $b_1, \ldots, b_n$ and $z_1, \ldots, z_n$ complex numbers satisfying (4) and $1 = |z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_n|$. Then

$$\max_{\nu}\mathrm{Re}\left(\sum_{j=1}^{n} b_j z_j^{\nu}\right) \geqslant \frac{1}{3n}\left\{\frac{n}{8e(m+n(3+\pi/\kappa))}\right\}^{2n}\min_{j=1,\ldots,n}|b_1 + \cdots + b_j|,$$

where the maximum is taken over all integers ν satisfying $m + 1 \leqslant \nu \leqslant m + n(3 + \pi/\kappa)$. Part I of the book, dealing with these kinds of minimax results, is concluded by a section with open problems. Part I covers about 200 pages and will be the standard source for Turán's power sum method from now on.

The fundamental character of Turán's main theorems becomes clear from the wide range of applications. Part II contains applications to function theory (50 pp.), differential equations (18 pp.), numerical algebra (22 pp.) and number theory (237 pp.). Turán's First Main Theorem has the remarkable property that it is a relation between function values. Putting $F(t) = \sum_{j=1}^{n} b_j z_j^t$, inequality (2) takes the form

$$\max_{t=m+1,\ldots,m+n} |F(t)| \geqslant \left(\frac{n}{2e(m+n)}\right)^n F(0).$$

A straightforward application of this inequality yields that any polynomial $P(z) := \sum_{j=1}^{n} b_j z^{m_j}$ with $0 \leqslant m_1 < m_2 < \cdots < m_n$ and any real numbers α, δ with $0 \leqslant \alpha < \alpha + \delta \leqslant 2\pi$ satisfy the inequality

$$\max_{|z|=1} |P(z)| \leqslant \left(\frac{4e\pi}{\delta}\right)^n \max_{\substack{z=1 \\ \alpha \leqslant \arg z \leqslant \alpha+\delta}} |P(z)|.$$

Note that the coefficient depends on the number of terms of P, but not on the degree. By employing such inequalities Turán proved Fabry's Gap Theorem and its extension to Dirichlet series, essentially due to Carlson-Landau and Szász. They were also used by Gaier in 1965 to prove the high-indices theorem for Borel-summability. Another version of (4) reads: Let A, B and C be real numbers with $A \leqslant B < C$. If $y(t)$ satisfies the differential equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = 0, \tag{5}$$

where the a_ν are complex constants such that all the roots α of the characteristic polynomial $z^n + a_1 z^{n-1} + \cdots + a_n$ satisfy $\operatorname{Re}\alpha \geqslant 0$, then

$$y(A) \leqslant \left(2e\frac{C-A}{C-B}\right)^n \max_{B \leqslant t \leqslant C} |y(t)|.$$

This enables one to derive some interesting inequalities on solutions of differential equations such as the heat equation [9]. The reviewer used it to prove that the number of zeros of the solution $y(z) = \sum_{k=1}^{l} P_k(z) e^{\omega_k z}$ of (5) in a disk of radius R is at most $3n + 4R\Omega$, where $\Omega = \max_k |\omega_k|$. Voorhoeve [**10**] obtained in a completely different way the upper bound $2n + 4R\Omega/\pi$. The applications of these results and the related results on small values to transcendental number theory (cf. Tijdeman [7], Chudnovsky [**4**], Brownawell [3]), are not worked out in Turán's book.

The other applications to number theory deal almost entirely with prime number theory. Riemann stated in 1859 without proof that $\Delta(x) := \pi(x) - \operatorname{li} x$ satisfies $\Delta(x) < 0$ for $x > 2$. In 1914 Littlewood [**5**] showed that $\Delta(x)$ has infinitely many sign changes. Pólya [**6**] gave a lower bound for the number of sign changes of $\Delta(x)$. By using the Second Main Theorem this bound was later improved upon by Knapowski-Turán and Pintz. The latter showed for example that there exist effectively computable constants c_1 and c_2 such that for $X > c_1$ the number of sign changes of $\Delta(x)$ in $[0, X]$ exceeds $c_2\sqrt{\log X}/\log\log X$. A variant of the Second Main Theorem due to Knapowski and Turán's Third Main Theorem are used in a similar manner in the comparative prime number theory. In this theory the difference $\Delta_0(x) := \pi(x, q, l_1) - \pi(x, q, l_2)$ of primes $\leqslant x$ congruent to l_1 and congruent to $l_2 \bmod q$ is studied. In a long series of papers Knapowski and Turán have given lower bounds for the number of sign changes of $\Delta_0(x)$ and for the first sign change of $\Delta_0(x)$ for various values of l_1 and l_2 under suitable conditions, namely the Haselgrove condition and sometimes the finite Riemann-Piltz conjecture. For this kind of results Turán's main theorems seem to be the most natural tool. Many other applications have also been proved without using the main theorems. Among these applications are Siegel's theorem on L-functions, Linnik's theorem on the smallest prime in an arithmetical progression, and the conditional and unconditional upper bounds of Halász and Turán for the number of zeros of $\zeta(s)$ in the rectangle $\alpha \leqslant \operatorname{Re} s \leqslant 1$, $|\operatorname{Im} s| \leqslant T$.

If some other author would have written this book, he would probably have called it "Turán's Power Sum Method". It is the monumental result of the labour of one of the leading number theorists of the period 1950–1975. The book raises the question how far the personal impact of a mathematician influences the stream of mathematics. Would the main theorems be known at this time without the person and influence of Turán? Another question is whether it is desirable to write a book about a method and not to select to the type of results. I prefer this book above a volume of selected or collected papers. Moreover, certain parts of the book give an excellent account of a mathematical theory. Of course Part I does, together with the first sections of Part II. Further, the applications to the sign changes of $\pi(x) - \operatorname{li} x$ and to comparative number theory provide an almost complete treatment of the present theory. Together this covers two thirds of the book. I think that one can only measure the importance of the other applications by comparing them with the existing standard works dealing with these applications. In Turán's book the references to related results proved by other methods are sparse, and recent references are often missing. However, for experts in function theory, numerical analysis, or analytic number theory this book gives a good idea of what can be proved by the power sum method. Besides, this book with its simple basic theorems, its various applications, and its open problems, is an excellent source for an advanced course. Its appearance will help mathematicians to understand the use of the main theorems in applications better and will lead to new interesting results.

References

1. H. Bohr, *Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$*, Gött. Nachr. (1911), 409–428.

2. ______, *Sur l'existence des valeurs arbitrairement petites de la fonction $\zeta(s) = \zeta(\sigma + it)$ de Riemann pour $\sigma > 1$*, Oversigt Vidensk. Selsk. Kobenhaven (1911), 201–208.

3. W. D. Brownawell, *On the development of Gelfond's method* , Number Theory Carbondale 1979 Proceedings, Lecture Notes in Math., vol. 751, Springer-Verlag, Berlin etc., 1979, pp. 18–44.

4. G. V. Chudnovsky, *Some analytic methods in the theory of transcendental numbers*; *Analytic methods in diophantine approximations* (Russian). Preprints IM-74-8 and 9, Ukrainian SSR Acad. Sciences, Kiev 1974, 100 pp.

5. J. E. Littlewood, *Sur la distribution des nombres premiers*, C. R. Acad. Sci. Paris **158** (1914), 1869–1872.
6. G. Pólya, *Über das Vorzeichen des Restgliedes im Primzahlsatz*, Gött. Nachr. (1930), 19–27.
7. R. Tijdeman, *An auxiliary result in the theory of transcendental numbers*, J. Number Theory **5** (1973), 80–94.
8. P. Turán, *On a property of the stable or conditionally stable solutions of systems of nonlinear differential equations*, Ann. Math. **48** (1959), 333–340.
9. ______, *A remark on the heat equation*, J. Analyse Math. Jérusalem **14** (1965), 443–448.
10. M. Voorhoeve, *Zeros of exponential polynomials*, Thesis, University of Leiden, 1977.

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Invariance theory, the heat equation, and the Atiyah-Singer index theorem, by Peter B. Gilkey, Mathematics Lecture Series, vol. 11, Publish or Perish, Inc., Wilmington, Delaware, 1984, viii + 349 pp., $40.00. ISBN 0-914098-20-9

The announcement of what became known as the "Atiyah-Singer Index Theorem" appeared in this Bulletin in 1963. The full proof came out in the Annals of Mathematics in a series of papers between 1968 and 1971, although *Seminar on Atiyah-Singer Index Theorem*, edited by R. Palais, was published in 1965 and contained discussions of the proof and the background needed to understand the theorem. This beginning was typical of the development of the subject. A tremendous amount of research has been generated, yet there has been a relative scarcity of sources for the uninitiated. The book under review is the first detailed exposition of the approach to the index theorem developed by its author and independently by V. K. Patodi in the early seventies. My main complaint about the book is that it is long overdue. All this is partly explained by the heavy demand posed by the subject on both the student and the expository writer. What is required is considerable breadth. Familiarity with analysis, algebraic topology, and Riemannian geometry is an absolute minimum for the student. The subject is in constant flux and has interacted with (this list is certainly incomplete and the ordering is random) number theory, algebraic geometry, mathematical physics, representation theory of Lie groups, probability, and Riemannian geometry.

Index theory is the study of global aspects of systems of linear elliptic partial differential equations. One considers an elliptic operator D between spaces of C^∞ sections of two hermitian vector bundles E and F on a compact Riemannian manifold M. The adjoint operator D^* is also elliptic and, because of the ellipticity, the spaces of solutions of the equations $Du = 0$ and $D^*u = 0$ are finite dimensional. The index of D, $\operatorname{ind}(D)$, is defined as

$$\operatorname{ind}(D) = \dim\ker D - \dim\ker D^*.$$

The celebrated index formula of Atiyah and Singer computes this index as an integral of a *locally* defined expression. More precisely, the integrand is a