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ON EXTENDING SOLUTIONS TO WAVE EQUATIONS ACROSS GLANCING BOUNDARIES

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Introduction. The purpose of this note is to announce some results on the following extension problem. On a C^∞ manifold M with boundary, if u is a given extendible distribution satisfying

$$Pu \in C^\infty(M), \tag{1}$$

under what conditions (on $P, \partial M$, and $u|_{\partial M}$) can u be extended across ∂M as a solution, that is, to a distribution $\tilde{u} \in D'(\tilde{M})$ such that $P\tilde{u} \in C^\infty(\tilde{M})$, for some open manifold $\tilde{M}$ extending M across ∂M? Here P is assumed to be a second-order differential operator on M with smooth coefficients, noncharacteristic with respect to ∂M, and with real principal symbol p having fiber-simple characteristics

$$d_{\text{fiber}}p \neq 0 \quad \text{on} \quad p^{-1}(0) \cap (T^*M\backslash 0) \tag{2}$$

(for example, the wave operator acting in the exterior of a smooth obstacle). After extending the coefficients of P smoothly across ∂M, we can view P as an operator on some open extension $\tilde{M}$ of M.

The problem is easily solved in the two cases where no null bicharacteristics tangent to ∂T^*M are present. When ∂M is everywhere elliptic with respect to P, classical theory implies that the desired $\tilde{u}$ can be found if and only if $u|_{\partial M} \in C^\infty(\partial M)$. When ∂M is everywhere hyperbolic, nothing has to be assumed about $u|_{\partial M} \in D'(\partial M)$, for the extension $\tilde{u}$ can be produced simply by solving the Cauchy problem in a neighborhood of ∂M with Cauchy data given by u. Here we are interested in the two cases where null bicharacteristics tangent to ∂T^*M to first order are present, the diffractive and gliding cases. An example given in [**8**] shows that if the boundary is diffractive, even when $u|_{\partial M}$ is smooth, it may happen that no extension as a solution (in fact, no extension $\tilde{u}$ such that $\rho \notin WF\,P\tilde{u}$ where $\rho \in \partial T^*M$ is a point of null bicharacteristic tangency) exists. Our main result (Theorem 2) implies that, in contrast to the diffractive case, near gliding points extensions as microlocal solutions always exist when $u|_{\partial M}$ is smooth. We construct such an extension after showing that, near a gliding point $\sigma \notin WF\,u|_{\partial M}$, any distribution u satisfying $Pu \in C^\infty(M)$ has the series expansion given in Theorem 1. The proof of Theorem 2 makes essential use of the recent unified treatment of the diffractive and gliding parametrices [**5**], in which the eikonal and transport equations are solved on both sides of the boundary. Full proofs will appear in [**9**].

We proceed to recall some terminology.

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Boundary points. Let $\iota\colon \partial M \to M$ be the inclusion inducing the projection $\iota^*\colon \partial T^*M \to T^*\partial M$. Then the elliptic, hyperbolic, and glancing regions are respectively

$$E = \{\sigma \in T^*\partial M\backslash 0 \colon p \neq 0 \text{ on the line } {\iota^*}^{-1}(\sigma)\}.$$
$$H = \{\sigma \in T^*\partial M\backslash 0 \colon p \text{ has (two) simple zeros in } {\iota^*}^{-1}(\sigma)\}.$$
$$G = \{\sigma \in T^*\partial M\backslash 0 \colon p \text{ has a double zero, } \rho, \text{ in } {\iota^*}^{-1}(\sigma)\}.$$

Letting x be a real C^∞ function vanishing simply on ∂M and positive in $\mathring{M}$ near ∂M, we write $G = G_d \cup G_g \cup G_0$ (a union of the diffractive, gliding, and higher-order tangency regions), where $\sigma \in G_d$, G_g, or G_0 depending on whether $\{p, \{p, x\}\}(\rho) > 0$, < 0, or $= 0$ respectively.

Gliding parametrices. Let $\sigma = \iota^*(\rho) \in G_g$. Choose coordinates $(x, y, \xi, \eta) \in T^*\mathbf{R}^{n+1}$ such that $\mathring{M} = \{x > 0\}$, $\rho = (0, 0, 0, \overline{\eta})$, and $H_p y_1(\rho) > 0$ (so y_1 will serve as our time variable). The forward and backward gliding parametrices at σ are maps $B_\pm\colon D'(\partial M) \to D'(M)$ such that for some small neighborhood $U \subset M$ of $\pi\sigma$ and some small conic neighborhood $\Lambda \subset T^*\partial M\backslash 0$ of σ, we have $PB_\pm g \in C^\infty(U)$ and $B_\pm g|_{\partial M} = g \bmod C^\infty(\partial M)$ for all g with $WF\, g \subset \Lambda$. Moreover, B_+ (resp. B_-) propagates singularities in the direction of increasing (resp. decreasing) $y_1 \cdot B_\pm$ (see [**2**, **5**, or **7**]) are constructed from Fourier-Airy integral operators

$$C_\pm F(x, y) = (2\pi)^{-n} \int_{\Gamma_\pm} e^{i\phi(x,y,\mu)} I(a, b, \varsigma; x, y, \mu) \hat{F}(\mu)\, d\mu, \tag{3}$$

where

$$I = [a(x, y, \mu) \operatorname{Ai}(\varsigma(x, y, \mu)) + b(x, y, \mu) \operatorname{Ai}'(\varsigma(x, y, \mu))]/\operatorname{Ai}(\varsigma_0(\mu)).$$

Here $\mu = (z, \mu') = (z, \mu_2, \ldots, \mu_n) \in \mathbf{C} \times \mathbf{R}^{n-1}$ and $\operatorname{Im} z = \mp T$ $(T > 0)$ on $\Gamma_\pm$ respectively. Ai is the standard Airy function (see [**6**, p. 218]), an entire function whose zeros are all simple and negative. $\varsigma_0(\mu) = \varsigma(0, y, \mu) = z\mu_n^{-1/3}$, so $(\operatorname{Ai}(\varsigma_0))^{-1}$ makes sense on $\Gamma_\pm$. The phase functions ϕ, ς and the symbols a, b are obtained by taking almost analytic extensions (see [**3**]) in the μ_1 variable of functions solving eikonal and transport equations on both sides of $x = 0$. The symbols a and b are supported in a small conic neighborhood $\nu \subset \{(x, y, \mu)\colon \mu_n \geq C|\mu|\}$ of $(0, 0, \overline{\mu})$, where $\overline{\mu} = (0, \ldots, 0, 1)$. Finally, we recall that ϕ and b also satisfy $\phi'_{x,y}(0, 0, \overline{\mu}) = (0, \overline{\eta})$ and $b|_{x=0} = 0$.

Operators $B_\pm$ with the desired properties can now be obtained by setting $B_\pm = C_\pm J$, where $J\colon E'(\partial M) \to E'(\mathbf{R}^n)$ is a proper elliptic F.I.O. microlocally inverting the boundary operators $(C_\pm)|_{x=0}$.

Note that although ϕ, ς, a, and b are defined on both sides of $x = 0$, the operators $B_\pm$ are defined *only* in $x \geq 0$ because the Airy quotients in (3) blow up exponentially in $x < 0$.

Main results. We number the zeros r_k of $\operatorname{Ai}(z)$ so that $0 > r_1 > r_2 > \cdots \to -\infty$.

THEOREM 1. *Let P be a second-order differential operator on M non-characteristic with respect to ∂M, with real principal symbol p satisfying* (2). *If $\sigma \in G_g$ and $u \in D'(M)$ satisfies $Pu \in C^\infty(M)$ and $\sigma \notin WF\, u|_{\partial M}$, then $u = v_1 + v_2$, where $\sigma \notin WF_b v_2$ (WF_b is defined in* [**4**]) *and $v_1 = \sum_k u_k =$*

$$i(2\pi)^{1-n}\sum_k \int e^{i\phi(x,y,\tilde{\mu}_k)}\alpha_k \mu_n^{1/3}[(a\,\mathrm{Ai}(\varsigma)+b\,\mathrm{Ai}'(\varsigma))(x,y,\tilde{\mu}_k)]\hat{F}(\tilde{\mu}_k)\,d\mu_2\cdots d\mu_n.$$

Here $\tilde{\mu}_k = (r_k\mu_n^{1/3}, \mu_2, \ldots, \mu_n)$, α_k is the residue of $(\mathrm{Ai}(z))^{-1}$ at r_k, a, b, ϕ, ς are as in (3), *and $F \in E'(\mathbf{R}^n)$.*

Since $b(0,y,\mu) = 0$ and $\varsigma(0,y,\mu) = z\mu_n^{-1/3}$, each of the terms u_k satisfies $u_k|_{x=0} = 0$ as well as $Pu_k \in C^\infty(U)$ for some neighborhood $U \subset M$ of $\pi\sigma$. The factors $(\mathrm{Ai}(\varsigma_0))^{-1}$ which forced us to consider only $x \geq 0$ when defining $B_\pm$ have now disappeared, so the fact that ϕ, ς, a, and $b(x, y, \tilde{\mu}_k)$ satisfy eikonal and transport equations in a two-sided neighborhood $\tilde{U} \subset \tilde{M}$ of $\pi\sigma$ can be put to use. We deduce that each u_k extends to a $\tilde{u}_k \in D'(\tilde{U})$ such that $P\tilde{u}_k \in C^\infty(\tilde{U})$. This suggests

THEOREM 2. *Let P and M be as in Theorem* 1. *If $\sigma \in G_g$ and if $u \in D'(M)$ satisfies $Pu \in C^\infty(M)$ and $\sigma \notin WF\, u|_{\partial M}$, then an extension $\tilde{u}$ can be constructed such that $WF\, P\tilde{u} \cap \iota^{*^{-1}}(\Gamma) = \emptyset$, for some conic neighborhood $\Gamma \subset T^*\partial M\backslash 0$ of σ.*

Sketch of the proofs. Using the fact that regularity propagates in the boundary near gliding points (see [**1**]), we first find a distribution $g \in E'(\partial M)$, supported in $y_1 < 0$, for which $u = v_1 + v_2$ where $v_1 = B_+g - B_-g$ and $\sigma \notin WF_b v_2$. Putting $F = Jg$ we have $(2\pi)^n(B_+g - B_-g) =$

$$\int_{\Gamma_+} e^{i\phi} I(a,b,\varsigma)\hat{F}(z,\mu')\,dz\,d\mu' - \int_{\Gamma_-} e^{i\phi} I\hat{F}\,dz\,d\mu'. \tag{4}$$

We compute the integral with respect to z first (noting that the integrand has poles at $r_k\mu_n^{1/3}$, $k = 1, 2, \ldots$) by taking the limit as $k \to \infty$ of integrals around closed rectangular contours in $\mathbf{C}$ centered at the origin. In [**5**] it is shown that the zeros σ_k of $\mathrm{Ai}'(z)$ satisfy $0 > \sigma_1 > r_1 > \sigma_2 > r_2 \cdots \to -\infty$. So it is convenient to make the left vertical side of the kth rectangle pass through $\sigma_{k+1}\mu_n^{1/3}$. Estimates of the Airy quotients on the vertical segments show that the contributions to the contour integrals from those segments approach zero as $k \to \infty$. Thus Cauchy's integral formula with remainder yields that (4) equals

$$\begin{aligned}(5)\quad 2\pi i \sum_k \int e^{i\phi(x,y,\tilde{\mu}_k)}\alpha_k\mu_n^{1/3}[(a\,\mathrm{Ai}(\varsigma) + b\,\mathrm{Ai}'(\varsigma))(x,y,\tilde{\mu}_k)]\hat{F}(\tilde{\mu}_k)\,d\mu' \\ + \iint_W \partial_{\bar{z}}(e^{i\phi}I(a,b,\varsigma)\hat{F})\,d\bar{z}\wedge dz\,d\mu',\end{aligned}$$

where $W = \{z \in \mathbf{C}\colon -T \leq \operatorname{Im} z \leq T\}$. Though the integrand in the second term has infinitely many poles in W, estimates of $(\mathrm{Ai}(\varsigma_0))^{-1}$ near the negative real axis and the fact that ϕ, ς, a, and b are almost analytic in z imply that the second term is smooth in $x \geq 0$. This finishes the proof of Theorem 1.

As noted above, each u_k extends to a $\tilde{u}_k \in D'(\tilde{U})$ such that $P\tilde{u}_k \in C^\infty(\tilde{U})$. Let $\chi(y) \in C_0^\infty(\mathbf{R}^n)$. Then repeated integrations by parts with respect to the y variable yield, for all $N > 0$, the estimates $|\langle \tilde{u}_k(x, \cdot), \chi(\cdot)\rangle| \leq C_N|\alpha_k|\,|r_k|^{-N}$ with C_N independent of k. Similar estimates clearly hold for $\partial_x^\beta \tilde{u}_k$ as well. Since $r_k \sim -ck^{2/3}$ (see [**5**, Appendix A]) and since the residues α_k can be shown to satisfy $|\alpha_k| \leq C|r_k|^{-1/4}$, we may conclude that the sum $\sum_k \tilde{u}_k$ converges to a distribution $\tilde{v}_1 \in C^\infty(R\colon D'(\mathbf{R}^n))$ extending v_1. Moreover, for all $N > 0$, $|P\tilde{u}_k| \leq C_N|r_k|^{-N}$ in $\tilde{U}$ (by the properties of ϕ, ς, a, and b), and similar estimates hold for $\partial_{x,y}^\beta(P\tilde{u}_k)$. Hence $P\tilde{v}_1 = \sum_k P\tilde{u}_k \in C^\infty(\tilde{U})$. To complete the proof of Theorem 2, we invoke a simple lemma [**8**, 1.6] which implies that since $\sigma \notin WF_b v_2$, an extension $\tilde{v}_2$ of v_2 can be constructed such that $WF\,\tilde{v}_2 \cap {\iota^*}^{-1}(\Gamma) = \emptyset$ for some conic neighborhood $\Gamma \ni \sigma$. So $\tilde{u} = \tilde{v}_1 + \tilde{v}_2$ is the desired extension of u.

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