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FOLIATIONS AND SURGERY ON KNOTS

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In this note we will first discuss some of the properties of 3-manifolds which possess taut (defined below) foliations. Next we will describe our main results which assert that many surgered 3-manifolds possess taut foliations. Finally we will show how these existence theorems together with the previously stated results can be exploited to produce topological corollaries, e.g., knots in $S^2 \times S^1$ not contained in 3-cells are determined by their complements, and knots in S^3 have property R.

For clarity and brevity many of the results are not stated in full generality and the discussion of smooth versions of the announced foliations results is omitted. Details to all the results can be found in [**G2**, **G3**, and **G4**].

Let $\mathcal{F}$ be a transversely oriented codimension one foliation on a compact oriented 3-manifold M such that $\mathcal{F}$ is transverse to ∂M. $\mathcal{F}$ is *taut* if for each leaf L of $\mathcal{F}$ there exists a curve γ transverse to $\mathcal{F}$ such that $\gamma \cap L \neq \emptyset$.

The existence of such a taut $\mathcal{F}$ implies that ∂M is a (possibly empty) union of tori and M is either $S^2 \times S^1$ (and $\mathcal{F}$ is the product foliation) or M is irreducible. The first condition follows by Euler characteristic reasons, while the latter follows from the work of Reeb [**R**], Haefliger, Novikov [**N**], and Alexander [**A**]. See also [**Ro**]. [Historical note: Armed with the work of [**R** and **N**] the reader will see in [**A**] a proof of the fact that a separating 2-sphere in a tautly foliated 3-manifold bounds a 3-cell. This proof was rediscovered in [**Ro**]]. Novikov also showed that a curve transverse to $\mathcal{F}$ is homotopically of infinite order. Thurston [**T**] has shown that any compact leaf L is a Thurston norm-minimizing surface [i.e., $|\chi(L')| \leq |\chi(T')|$ for any properly embedded T with $[T] = [L] \in H_2(M, \partial M)$, where S' denotes $S-$(sphere and disc components)] for the class $[L] \in H_2(M, \partial M)$. Geometrically (if $\mathcal{F}$ is also C^2) Sullivan [**S**] has shown that M possesses a Riemannian metric such that each leaf is minimal, i.e., locally area minimizing. In fact by Harvey and Lawson $\mathcal{F}$ can be calibrated [**HL**].

DEFINITIONS. Let M be a 3-manifold such that ∂M contains a torus T. N is said to be obtained by *Dehn filling* M along an essential simple closed curve γ in T, if N is obtained by first attaching a 2-handle to M along γ and then capping off the resulting 2-sphere with a 3-cell. N is obtained by attaching a solid torus, called the *filling*, to M and $M = N - \mathring{N}(k)$ where k is the *core* of the filling. Note that a manifold obtained by Dehn surgery on a knot $k \subset N$ is a manifold obtained by Dehn filling $N - \mathring{N}(k)$. A manifold

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M is *atoroidal* if every embedded incompressible (i.e., injects on π_1) torus is boundary parallel.

A leaf of a foliation is of *depth* 0 if it is compact. Having defined the depth $< p$ leaves we say that a leaf is depth p if it is proper (i.e., the subspace topology on L equals the leaf topology), L is not of depth $< p$ and $\overline{L} - L \subset$ (union of depth $< p$ leaves). If the depth of each leaf is defined and bounded above, then F is a *finite-depth* foliation and the depth of the foliation is the depth of the maximal depth leaf. A finite-depth foliation is topologically very tame although its transverse structure may be C^0.

THEOREM 1. *Let M be an atoroidal Haken 3-manifold whose boundary is a torus and $H_2(M) \neq 0$. Let S be any Thurston norm-minimizing surface representing a class of $H_2(M)$. Then with at most one exception (up to isotopy) the manifold N obtained by filling M along an essential simple closed curve in ∂M possesses a taut finite-depth foliation $\mathcal{F}$ such that S is a leaf of $\mathcal{F}$ and the core of the filling is transverse to $\mathcal{F}$.*

THEOREM 2. *Let S be a minimal genus Seifert surface for a knot k in S^3. There exists a taut finite-depth foliation $\mathcal{F}$ of $S^3 - \mathring{N}(k)$ such that S is a leaf of $\mathcal{F}$ and $\mathcal{F} \mid \partial N(k)$ is a foliation by circles.*

REMARK. In [**G1**] it was shown that if S is a minimal genus Seifert surface for a knot k in S^3, then there exists a taut finite-depth foliation $\mathcal{G}$ of $S^3 - \mathring{N}(k)$ such that S is a leaf of $\mathcal{G}$. In this case $\mathcal{G} \mid \partial N(k)$ is the suspension of a map of the circle. The improvement is that $\mathcal{F} \mid \partial N(k)$ contains only compact leaves, while $\mathcal{G} \mid \partial N(k)$ might have contained noncompact leaves.

DEFINITION. A knot k is *determined by its complement* in the oriented 3-manifold M if, for any knot $k' \subset M$, there exists an orientation-preserving homeomorphism $f\colon (M, k) \to (M, k')$ if and only if there exists an orientation-preserving homeomorphism between $M - k$ and $M - k'$. It follows that a knot k is determined by its complement if every nontrivial surgery on k yields a manifold distinct from M.

COROLLARY 3. *If k is a knot in $S^2 \times S^1$ which does not lie in a 3-cell, then k is determined by its complement.*

IDEA OF PROOF. We give the proof in the case $S^2 \times S^1 - \mathring{N}(k)$ contains no essential tori. We will show that nontrivial surgery on k never yields $S^2 \times S^1$.

If k is homologically nontrivial, then nontrivial surgery on k does not yield a homology $S^2 \times S^1$.

If k is homologically trivial, then $H_2(S^2 \times S^1 - \mathring{N}(k)) \neq 0$, so some closed Thurston norm-minimizing surface S exists in $S^2 \times S^1 - \mathring{N}(k)$. S is not a 2-sphere else k lies in a 3-cell. Apply Theorem 1 to conclude that (with at most one exception) any surgery on k yields a manifold possessing a taut foliation with S as a leaf. By Alexander and Novikov such a surgered manifold is irreducible. We conclude that the exceptional surgery is the trivial one.

COROLLARY 4. *Let M be a Haken atoroidal 3-manifold such that ∂M is a torus. Let S be any closed Thurston norm-minimizing surface. With at most* *one exception (up to isotopy) the following holds. If N is obtained by filling M along an essential curve $\alpha \subset \partial M$, then*

(1) *S is norm-minimizing in N,*
(2) *S is incompressible in N,*
(3) *the core of the filling is of infinite order in $\pi_1(N)$, and*
(4) *N is irreducible.*

PROOF. Apply Theorem 1 to S and M to conclude that if N is obtained by doing a nonexceptional filling to ∂M, then N possesses a taut foliation $\mathcal{F}$ with S as a leaf. α can be viewed as the core of the solid torus which one attaches to ∂M to obtain S, so by Theorem 1, α is transverse to $\mathcal{F}$. By Thurston (1) holds and in particular (2) holds. By Novikov (3) and (4) hold.

REMARKS. By Thurston, any homology class can be represented by a norm-minimizing surface, and closed norm-minimizing surfaces in hyperbolic 3-manifolds satisfy the hypotheses of Corollary 4.

DEFINITIONS. Let k be a knot in a closed oriented 3-manifold N. A *longitude* of k is the unique (up to isotopy) essential simple closed curve in $\partial N(k)$ such that $0 = [\lambda] \in H_2(N - \overset{\circ}{N}(k), \mathbf{Q})$. M is obtained by *zero-frame surgery* on a knot k in N, if it is obtained by performing Dehn surgery to the longitude. Note that if N is a homology sphere, then M is the unique manifold obtained by Dehn surgery on k which is a homology $S^2 \times S^1$.

COROLLARY 5. *If M is obtained by performing zero-frame surgery on a knot k in S^3, then M is prime and genus $k = \min\{$genus $S \mid S$ is an embedded oriented nonseparating surface in $M\}$.*

PROOF. Let R be a minimal genus surface for k. Apply Theorem 2 to find a taut foliation $\mathcal{F}'$ of $S^3 - \overset{\circ}{N}(k)$ such that $\mathcal{F}'|\partial N(k)$ is a foliation by longitudes and R is a leaf. Cap off the leaves of $\mathcal{F}'|\partial N(k)$ by discs to extend $\mathcal{F}'$ to a taut foliation $\mathcal{F}$ on M. Cap off R by a disc to create the surface T. Genus $k =$ genus $R =$ genus T and T generates $H_2(M)$. By Thurston genus $T = \min\{\text{genus } S \mid S \text{ is an embedded oriented nonseparating surface in } M\}$. By Novikov, Alexander, and Reeb, M is prime.

REMARK. The *property R* conjecture asserts that nontrivial surgery on a nontrivial knot k in S^3 does not yield $S^2 \times S^1$, i.e.,

$$\{\text{the `trivial' knot complement}\} = \{\text{complements of knots in } S^2 \times S^1\} \cap \{\text{complements of knots in } S^3\} = D^2 \times S^1.$$

Since zero-frame surgery on a knot in S^3 is the only way to obtain a homology $S^2 \times S^1$, to prove property R it suffices to show that zero-frame surgery on a nontrivial knot k in S^3 does not yield $S^2 \times S^1$. The *Poenaru* conjecture asserts that zero-frame surgery on a nontrivial knot k in S^1 does not yield a $S^2 \times S^1 \# M^3$. Corollary 5 gives positive proofs of these conjectures.

The most striking observation in the proof of Theorem 2 is that any nice finite-depth taut partial foliation constructed on $S^3 - \overset{\circ}{N}(k)$ extends to a foliation satisfying the conclusions of that result. This is the key ingredient in proving

COROLLARY 6. *k is a fibered knot of genus g in S^3 if and only if the manifold M obtained by performing zero-frame surgery to k fibers over S^1 with fiber a surface of genus g.*

REMARK. By Gonzalez-Acuña [**Ga**] the trefoil and the figure 8 knots are the only genus one fibered knots in S^3. Therefore surgery on a knot k in S^3 yields a torus bundle over S^1 if and only if either k is the trefoil knot or k is the figure 8 knot.

By applying Theorem 1 and a combination of the proofs of Theorem 2 and Corollary 5 to a knot embedded in a solid torus we obtain the following result (whose discovery was stimulated by a discussion with Marty Scharlemann).

COROLLARY 7. *Let k be a knot in $D^2 \times S^1$ whose geometric intersection number with a $D^2 \times \mathrm{pt}$ is nonzero. If M is a manifold obtained by nontrivial surgery on k, then one of the following must hold.*

(1) $M = D^2 \times S^1$. In this case k is a braid (from the point of view of both the original $D^2 \times S^1$ and M).

(2) M is irreducible and ∂M is incompressible.

(3) $M = M' \# W$, where W is a closed 3-manifold and $H_1(W)$ is finite and nontrivial.

A knot k in S^3 satisfies *property P* if it is either the trivial knot or only the trivial surgery on k yields a homotopy 3-sphere.

COROLLARY 8. *If k is a knot in S^3 such that $S^3 - \overset{\circ}{N}(k)$ contains either an essential annulus or torus, then k satisfies property P.*

PROOF. Such a k is either a torus knot or there exists a torus $T \subset S^3$ such that one side of T bounds a $D^2 \times S^1$ which contains k and the other side of T is a 3-manifold N with incompressible boundary. By [**Mo**] property P holds for torus knots.

If k is not a torus knot, then a manifold H obtained by nontrivial surgery on k is a union of N and a manifold M (as in the conclusion of Corollary 7) glued along T. If M is of type (3), then $\pi_1(H)$ is nontrivial. If M is of type (2), then T is incompressible in H so again $\pi_1(M)$ is nontrivial. If M is of type (1), then H is obtained by attaching a solid torus to N, so H is obtained by a surgery on a companion knot k' to k. A calculation [**Go**] yields the fact that either $H_1(H)$ is nontrivial or H was obtained by a surgery of type $1/n\omega^2$ where ω is the winding number of k in $D^2 \times S^1$ and $n \in \mathbf{Z}$, $n \neq 0$. Since T was an essential torus and k is a braid, $\omega > 1$. Therefore $|n\omega| \geq 2$. By [**CGLS**], H is not a homotopy sphere.

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