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The bidual of $C(X)$. I, by S. Kaplan, Mathematics Studies, vol. 101, North-Holland, Amsterdam, The Netherlands 1984, xvi + 424 pp., $57.75 US/Dfl. 150.00. ISBN 0-444-87631-6

A *Riesz space* is a (real) linear space E endowed with a partial ordering $\leqslant$ which is translation-invariant (i.e. $x \leqslant y \Rightarrow x + z \leqslant y + z$) and a lattice (i.e. $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all x and y), and such that $\alpha x \geqslant 0$ whenever $x \geqslant 0$ in E and $\alpha \geqslant 0$ in $\mathbf{R}$. Write $E^+ = \{x\colon x \geqslant 0\}$. A *Riesz norm* on E is a norm $\| \ \|$ such that $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$, where $|x| = x \vee (-x)$. A *Banach lattice* is a Riesz space with a Riesz norm under which it is complete.

From the beginnings of functional analysis it has been recognized that many of the most important normed spaces are endowed naturally with Riesz space structures. The interactions of the three aspects of a Banach lattice—its linear, metric and order structures—lead to a rich and delightful, if not particularly deep, tapestry of interwoven motifs. We can study these either in the general, setting up an abstract theory, or in the particular, concentrating on well-known spaces of special importance. The book under review takes the latter course, though fully committed, in language and spirit, to the wider theory of normed Riesz spaces.

An *M-space* is a Banach lattice E in which $\|x \vee y\| = \max(\|x\|, \|y\|)$ whenever $x, y \in E^+$; an *L-space* is a Banach lattice E in which $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E^+$. There are effective representation theorems for both classes. A Banach lattice is an M-space iff it is isomorphic, as normed Riesz space, to the space $C_0(X)$ of continuous real-valued functions vanishing at infinity on some locally compact Hausdorff space X; it is an L-space iff it is isomorphic to the space $L^1(X)$ of equivalence classes of integrable real-valued functions on some measure space X. Among the M-spaces we naturally wish to identify those corresponding to compact spaces X; these are precisely the M-spaces with a *unit* e such that, for any x, $\|x\| \leqslant 1$ iff $|x| \leqslant e$.

Corresponding to the rich internal structure of Riesz spaces is an appropriately elaborate theory of morphisms between them. If E and F are Riesz spaces, we say that a linear map $T: E \to F$ is *positive* if $Tx \geqslant 0$ whenever $x \geqslant 0$ and a *Riesz homomorphism* if $T(x \vee y) = Tx \vee Ty$ for all $x, y \in E$. A positive linear map $T: E \to F$ is *order-continuous* if $\inf T[A] = 0$ in F whenever $A \subseteq E$ is a nonempty, downwards-directed set with infimum 0 in E; it is *sequentially order-continuous* if $\inf_{n \in \mathbf{N}} Tx_n = 0$ in F whenever $\langle x_n \rangle_{x \in \mathbf{N}}$ is a decreasing sequence with infimum 0 in E. If E and F are Banach lattices, then every positive linear map from E to F is norm-continuous; questions concerning order-continuity therefore amount to a sharper enquiry into the nature of operators than questions about norm-continuity. The norm dual E' of a Banach lattice is precisely the space $E^{\sim}$ of differences of positive linear functionals; within this we can distinguish the subspace $E^{\times}$ of differences of order-continuous positive linear functionals. For any Riesz space E, $E^{\sim}$ is also a Riesz space, if we say that $f \leqslant g$ means that $f(x) \leqslant g(x)$ for every $x \in E^+$. If E is an M-space, then $E' = E^{\sim}$ is an L-space; if E is an L-space, then $E' = E^{\sim} = E^{\times}$ is an M-space with unit.

If E is a Riesz space, a linear subspace F of E is the kernel of a Riesz homomorphism with domain E iff it is *solid*, i.e., $x \in F$ whenever $y \in F$ and $|x| \leqslant |y|$. In this case there is a natural Riesz space structure on E/F, saying that $x^{\cdot} \leqslant y^{\cdot}$ iff $(x - y) \vee 0 \in F$. F is the kernel of an order-continuous Riesz homomorphism iff it is a *band*, i.e., a solid linear subspace such that $\sup A \in F$ whenever $A \subseteq F$ and $\sup A$ is defined in E. F is the set of values of a Riesz homomorphism with codomain E iff it is a *Riesz subspace* of E, i.e., a linear subspace such that $x \vee y \in F$ whenever $x, y \in F$. Part of the importance of the space $E^{\times}$ is that it is a band in $E^{\sim}$, and the canonical map from E to $(E^{\times})^*$, the algebraic dual of $E^{\times}$, is an order-continuous Riesz homomorphism from E to $E^{\times\times}$.

All the mathematics above is perfectly standard, though no two authors can agree on notation; it is dealt with once again in the opening chapters of Kaplan's book. But his main concern is with spaces of the form $C = C(X)$, where X is a compact Hausdorff space. C is an M-space with unit; its dual $C' = C^{\sim}$ is an L-space; and its bidual $C'' = C'^{\times} = C'^{\sim}$ is another M-space with unit. The point at which the study of C'' becomes more than the study of L-spaces and M-spaces comes when we examine the canonical embedding of C in C''. This gives rise to a special structure which is the subject of the book under review.

We now approach one of the test theorems of analysis. There is a canonical bijection $\mu \mapsto h_\mu$ from the space M of Radon measures on X to the positive cone C'^{+} of C', given by writing $h_\mu(x) = \int x\, d\mu$ for $x \in C$. I call this a test theorem because an author's attitude to it is likely to determine his whole treatment of the subject. The Bourbaki school made it a tautology by defining a Radon measure to be a member of C'. I think this was simply a blunder; a measure is an extended-real-valued countably additive function on a σ-algebra of sets, and sooner or later you have to come to terms with the things. Kaplan does not mention the theorem, and finds alternative paths through the thickets which surround it. I used to be sympathetic to such enterprises, but have since come to feel that they are largely misplaced ingenuity. My present view is that if you are studying topic A, and find deep results from topic B standing in your way, it will in the long run be worth learning B properly; not so much because you will learn useful facts, as because your intuition will be able to operate on an extra wavelength.

In fact, Kaplan's terminology, as well as his results, make it plain that he is fully conscious of the importance of the representation theorem. Consider, for instance, the following space of functions, which is properly given a prominent place in his account. U is the set of bounded real-valued functions on X which are *universally measurable*, i.e., μ-measurable for every $\mu \in M$. If $x \in U$, then (by definition) $\int x\, d\mu$ exists for every $\mu \in M$, so there is a member $\hat{x}$ of C'' defined by writing $\hat{x}(h_\mu) = \int x\, d\mu$ for every $\mu \in M$. The map $x \mapsto \hat{x}$: $U \to C''$ is an injective sequentially order-continuous Riesz homomorphism which is *uniferent*, i.e., the unit of C'' is the image of the unit of C. Accordingly U can be identified, as normed Riesz space, with its image $\hat{U}$ in C''. If $x \in C$ then $\hat{x}(f) = f(x)$ for every $f \in C'$; thus the embedding of U in C'' extends the canonical embedding of C.

Elements and subsets of U can now be considered in terms of the Riesz space structure of C''. For instance, write $\mathscr{I}\hat{C}$ for the set of suprema in C'' of nonempty order-bounded upwards-directed subsets of $\hat{C}$, the image of C in C''. Then $\mathscr{I}\hat{C}$ is precisely the set of images in C'' of bounded lower-semi-continuous real-valued functions on X. Similarly, elements of $\mathscr{D}\hat{C}$, the set of infima in C'' of nonempty order-bounded downwards-directed subsets of $\hat{C}$, correspond to bounded upper-semi-continuous functions on X. The set of Baire measurable functions on X corresponds to the sequential order-closure of $\hat{C}$ in C''; the set of Borel measurable functions to the sequential order-closure of $\mathscr{I}\hat{C} - \mathscr{I}\hat{C}$. An element φ of C'' belongs to $\hat{U}$ iff there are nonempty sets $A, B \subseteq C''$ such that A is upwards-directed, B is downwards-directed, $\sup A = \inf B = \varphi$, and $[\alpha, \beta] \cap \hat{C} \neq \varnothing$ for every $\alpha \in A$ and $\beta \in B$; $\hat{U}$ can also be characterized as $\mathscr{D}\mathscr{I}\hat{C} \cap \mathscr{I}\mathscr{D}\hat{C}$. $C'' = \mathscr{D}\mathscr{I}\hat{U} = \mathscr{I}\mathscr{D}\hat{U}$.

A reverse approach starts from an arbitrary $\varphi \in C''$. Define φ^*: $X \to \mathbf{R}$ by writing $\varphi^*(t) = \varphi(\hat{t})$ for every $t \in X$, where $\hat{t} \in C'$ is defined by writing $\hat{t}(x) = x(t)$ for every $x \in C$. Then $\varphi \mapsto \varphi^*$: $C'' \to \ell^\infty(X)$ is an order-continuous uniferent Riesz homomorphism. If $x \in U$, then $\hat{x}^* = x$; that is to say, $\int \varphi^*\, d\mu$ exists and is equal to $\varphi(h_\mu)$ whenever $\varphi \in \hat{U}$ and $\mu \in M$. Next, for $\varphi \in C''$, set

$$u(\varphi) = \inf\{\hat{x}: x \in C,\ \hat{x} \geqslant \varphi\} \in C'',\ \delta(\varphi) = u(\varphi) + u(-\varphi).$$

Both $u(\varphi)$ and $\delta(\varphi)$ belong to $\hat{U}$, and $u(\varphi)^*$ and $\delta(\varphi)^*$ are upper-semi-continuous; $\varphi \in \hat{C}$ iff $\delta(\varphi)^* = 0$. The map $\varphi \mapsto u(\varphi)$: $C'' \to C''$ is a kind of closure operator; it is norm-continuous.

Let μ be a Radon measure on X, and $x \in \ell^\infty(X)$. We say that x is Riemann integrable with respect to μ if

$$\sup\left\{\int y\, d\mu: y \in C,\ y \leqslant x\right\} = \inf\left\{\int z\, d\mu: z \in C,\ z \geqslant x\right\};$$

in this case, of course, both are equal to $\int x\, d\mu$. Let $E \subseteq C'$ be a band which separates the points of C. Write $M_E = \{\mu: \mu \in M,\ h_\mu \in E\}$ and $K_E = \{f: f \in E,\ f \geqslant 0,\ \|f\| = 1\}$. Suppose that $\varphi \in C''$. Then the following are equivalent: (i) there is an $x \in \ell^\infty(X)$ such that x is Riemann integrable with respect to μ, and $\int x\,d\mu = \varphi(h_\mu)$, for every $\mu \in M_E$; (ii) there is a $\psi \in C''$ such that $\psi \upharpoonright E = \varphi \upharpoonright E$ and $\delta(\psi)(f) = 0$ for every $f \in E$; (iii) $\varphi \upharpoonright K_E$ is $\mathfrak{T}_s(C',C)$-continuous.

Let $Ra \subseteq C''$ be the set of those φ such that $u(|\varphi|)^*$ is zero except on a meagre set. Then Ra is a norm-closed solid linear subspace of C''; its polar in C' is precisely $C^\times$. (Note that $C^\times = \{0\}$ in many of the most important elementary cases.) The quotient C''/Ra is an M-space with unit; let $\tilde{C}$ be the image of $\hat{C}$ in C''/Ra; because $\hat{C} \cap Ra = \{0\}$, $\tilde{C}$ is canonically isomorphic, as M-space, to $\hat{C}$ and C. The Riesz subspace $\mathscr{I}\tilde{C} \cap \mathscr{D}\tilde{C}$ of C''/Ra is Dedekind complete, so can be identified with the Dedekind completion of C.

I have not mentioned the multiplicative structure of C. But this is implicit in the Riesz space structure; every M-space with unit has a canonical multiplicative structure, and uniferent Riesz homomorphisms between such spaces are multiplicative. Thus there are multiplications on C'' and C''/Ra which are consistent with the natural multiplications on C and U.

From what I have written it should be clear that the structure (C, C'') is a happy hunting ground for anyone who enjoys multifaceted phenomena. I should like to conclude by remarking on three of the lines of enquiry suggested by this book. (a) Is there any sense in which we can say that U is the largest subspace of $\ell^\infty(X)$ which can be naturally identified with a subspace of $C''(X)$? It may be necessary to use concepts from mathematical logic to explain what "naturally identify" can properly mean. (b) The space X can be retrieved, up to homeomorphism, from the Riesz space C, and the L-space C' can be found from C'', being identifiable as $(C'')^\times$. But widely varying spaces X can give rise to identical C' spaces. Maharam's theorem gives a simple complete classification of L-spaces in terms of densities of principal bands; is there an easy way to pick out the C' spaces from this classification, and to what extent can we derive topological properties of X from the properties of C'? (c) Because C'' is an M-space with unit, it can be identified with $C(Z)$ for an essentially unique compact Hausdorff space Z, and the embedding of C in C'' corresponds to a continuous surjection $q\colon Z \to X$. Is there a useful direct topological construction of (Z, q) from X? Which aspects of the structure (C, C'') can be effectively developed in terms of the triple (X, Z, q)?

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Methods of bifurcation theory, by Shui-Nee Chow and Jack K. Hale, A Series of Comprehensive Studies in Mathematics, vol. 251, Springer-Verlag, New York, 1982, xv + 515 pp., $48.00. ISBN 0-387-90664-9

The book under review is a major treatise on analytical methods in bifurcation theory. The theory is developed in the context of a large variety of