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POTENTIAL THEORY FOR THE SCHRÖDINGER EQUATION

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Recently there has been a wave of results [**2, 4, 5, 11, 15, 16, 17**], on what is now referred to as the conditional gauge theorem. These works were inspired by [**1** and **6**]. We prove this result in greater generality than before and derive interesting new consequences. Let

$$A = \sum \frac{\partial}{\partial x^j}\left(a_{ij}(x)\frac{\partial}{\partial x^i}\right)$$

be a uniformly elliptic operator whose coefficients are bounded measurable functions on a bounded Lipschitz domain $D \subseteq R^d$. Define the Kato class K_d as the class of functions on D such that

$$\lim_{\alpha\downarrow 0}\sup_{x\in D}\int_{|x-y|<\alpha}\frac{|V(y)|}{|x-y|^{d-2}}\,dy = 0.$$

Our approach is to prove results about the operator $L = A + V$ by using known results for A and studying the probabilistic quantity, the conditional gauge.

In order to introduce the conditional gauge let $p(t,x,y)$ be the Green function for the parabolic equation $A = \partial/\partial t$ on $D\times(0,\infty)$. Let (X_t, P_x) be the diffusion, killed at the exit time $\tau_D = \inf\{t>0: X_t \in D\}$, whose transition density is $p(t,x,y)$. The analysis involves the diffusion X_t conditioned on its exit position. This conditioned diffusion, see [**10**], has transition density $p^z(t,x,y) = K_A(x,z)^{-1}p(t,x,y)K_A(y,z)$, where K_A is the kernel function for A on D, $x,y\in D$, $z\in\partial D$. We shall write $P_x^z(\cdot) = P_x(\cdot\,|\,X_{\tau_D} = z)$ and $e_V(t) = \exp\{\int_0^t V(x_s)\,ds\}$. The so-called gauge is the function on D, $F(1;x) \equiv E_x[e_V(\tau_D)]$ and the conditional gauge is defined on $D\times\partial D$ by $F(1;x,z)\equiv E_x^z[e_V(\tau_D)]$. Theorem 1 was first proven in [**12**] when $A=\Delta$, V is bounded and ∂D is C^2, later when $A=\Delta$, $V\in K_d$ and ∂D is $C^{1,1}$ in [**16**] and recently when $A=\Delta$, $V\in L^p$ for some $p>d/2$ and ∂D is Lipschitz in [**13**]. Our main result is the following.

THEOREM 1. *Suppose the uniformly elliptic*

$$A = \sum \frac{\partial}{\partial x^j}\left(a_{ij}(x)\frac{\partial}{\partial x^i}\right)$$

has bounded measurable coefficients, $V\in K_d$, and $D\subseteq R^d$ is bounded and Lipschitz. Then $F(1;x)<\infty$ for some $x\in D$ iff there is a positive constant c such that $c^{-1}\le F(1;x,z)\le c$, $(x,z)\in D\times\partial D$.

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SKETCH FOR PROOF OF THEOREM 1. The proof follows [**16**] and requires a Green function-kernel function estimate. Let then G_A be the Green function for A and the domain D. What is required are

(a)
$$\frac{G_A(x,y)K_A(y,z)}{K_A(x,z)} \le c[|x-y|^{2-d} + |y-z|^{2-d}]$$
for some positive constant c and $x, y \in D$, $z \in \partial D$, and

(b) $E_x^z \tau_D < \infty$, $x \in D$, $z \in \partial D$.

The first involves repeated use of known estimates on G_A in terms of the Newtonian potential, an inequality due to Carleson, the boundary Harnack principle, and Harnack chain arguments, all of which are valid for A by [**3**]. The second follows easily from (a).

One may also condition X_t to converge to an interior point $y \in D$ at the finite path life-time T. Then by proving (a), with all K_A's replaced with G_A's and letting $z \in D$, we have

THEOREM 2. *$F(1;x) < \infty$ for some $x \in D$ if and only if there exists a positive constant c such that for all $x, y \in D$*
$$c^{-1} \le F(1;x,y) = E_x^y[e_V(T)] \le c.$$

The next result involves the harmonic measures w_A and w_L.

THEOREM 3. *Suppose $F(1;x) < \infty$ for some $x \in D$. Then if $L = A + V$*

(1) $w_L^x(dz) = F(1;x,z)w_A^x(dz)$, $(x,z) \in D \times \partial D$,

(2) $G_L(x,y) = F(1;x,y)G_A(x,y)$, $x, y \in D$.

PROOF. We discuss (1). With some work it can be shown that the Feynman-Kac formula holds. That is, the solution to the Dirichlet problem $Lu = 0$ on D, $u = f$ on ∂D is
$$E_x[f(X_{\tau_D})e_V(\tau_D)] = \int_{\partial D} f(z)F(1;x,z)w_A^x(dz) = \int_{\partial D} f(z)w_L^x(dz).$$
Thus $F(1;x,z)w_A^x(dz) = w_L^x(dz)$. Equation (2) follows as in [**17**]. □

REMARK. If $F(1;x) < \infty$ one gets that w_A and w_L are simultaneously A_p-weights. [**8** and **10**] give conditions on A implying w_A is an A_p-weight relative to surface area.

We mention some consequences of Theorems 1 and 3 without proof. In general, when the gauge is finite, potential-theoretic results that hold for A and depend on bounds for w_A and G_A will also hold for $L = A + V$. Theorem 4 was also proven in [**4**].

THEOREM 4 (HARNACK'S INEQUALITY). *Assume $F(1;x) < \infty$ for some $x \in D$. There exist positive constants r_0 and c such that if $r < r_0$ and $B(x_0, 2r) \subset D$, then for every positive solution to $Lu = 0$ in D we have*
$$u(x) \le cu(y), \qquad x, y \in B(x_0, r).$$

REMARK. Harnack's inequality holds for A by [**14**].

THEOREM 5 (BOUNDARY HARNACK PRINCIPLE). *Assume* $F(1;x) < \infty$ *for some* $x \in D$. *There exist positive constants* r_0 *and* c *such that if* $r < r_0$ *and* $z \in \partial D$ *then whenever* $Lu = Lv = 0$ *in* D, *and* u, v *are positive and vanish continuously on* $\partial D \cap B(z, 2r)$, *we have*

$$\frac{u}{v}(x) \leq c\frac{u}{v}(y), \qquad x, y \in B(z, r) \cap D.$$

REMARK. The boundary Harnack principle holds for A by [**3**].

THEOREM 6 (COMPARISON OF SOLUTIONS FOR A AND L). *Suppose* $F(1;x) < \infty$ *for some* $x \in D$. *There exist positive constants* r_0 *and* c *such that for any* $z \in \partial D$ *and* $r < r_0$ *if* u *and* f *are positive solutions* $Lu = 0$, $Af = 0$ *on* D *and vanish continuously on* $\partial D \cap B(z, 2r)$ *then*

$$\frac{u(x)}{f(x)} \leq c\frac{u(y)}{f(y)}, \qquad x, y \in B(z, r) \cap D.$$

THEOREM 7 (MARTIN REPRESENTATION). *If* $F(1;x) < \infty$ *for some* $x \in D$ *then the Martin boundary for* L *on* D *is* ∂D *and every positive solution to* $Lu = 0$ *in* D *has the representation*

$$u(x) = \int_{\partial D} K_L(x, z)\mu(dz)$$

where $K_L(x, z) = (F(1; x, z)/F(1; x_0, z))K_A(x, z)$ *and* $K_A(x_0, z) = 1$.

THEOREM 8 (REGULARITY OF BOUNDARY POINTS). *Suppose* $F(1;x) < \infty$ *for some* $x \in D$. *Then* $z \in \partial D$ *is regular for* L *whenever it is regular for* A.

REMARK. This uses (2) of Theorem 3. By results of [**13**] $z \in \partial D$ is regular for A if and only if it is regular for Δ. Thus when $F(1;x) < \infty$, Δ and L have the same regular points.

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