A NOTE ON CALDERÓN-ZYGMUND SINGULAR INTEGRAL CONVOLUTION OPERATORS

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The purpose of this note is to show that the notion of weak maximal function introduced in [1] (see also [4], where a similar notion is considered) can be used to obtain some new information on the Calderón-Zygmund singular integral convolution operator.

We will follow the notations of [3]. Let $K$ be a kernel in $\mathbb{R}^n$ of class $C^1$ outside the origin satisfying

1. $|K(x)| \leq C|x|^{-n},$
2. $|\nabla K(x)| \leq C|x|^{-n-1}.$

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty,$ set

$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} f(x-y)K(y)\,dy$

and

$T(f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon(f)(x), \quad T^*(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|.$

We will assume that $K$ satisfies the usual properties ensuring that the mapping $f \mapsto T^*(f)$ is of weak type $(1,1)$ and that $T(f)(x)$ makes sense for a.e. $x.$

The notation $L^{1,\infty}$ will stand for the space of weak $L^1$ functions, and if $\varphi \in L^{1,\infty}$ and $B$ is a ball we write

$|||\varphi|||_{1,\infty}^B = \sup_{\delta > 0} \delta m(\{x \in B: |\varphi(x)| > \delta\})$

for the weak $L^1$ “norm” of $\varphi$ on $B.$ If $B = \mathbb{R}^n,$ we simply write $|||\varphi|||_{1,\infty}.$

The weak maximal function introduced in [1] is defined for $\varphi \in L^{1,\infty}$ by

$M_w\varphi(x) = \sup_{x \in B} \frac{|||\varphi|||_{1,\infty}^B}{m(B)},$

the supremum being taken over all balls centered at $x.$ The notation $M_w^m\varphi$ stands for the function obtained by applying $m$ times the operator $M_w,$ whenever this makes sense. In [1] it was already pointed out that for any $m$ there is a $\varphi \in L^{1,\infty}$ such that $M_w^j\varphi \in L^{1,\infty}$ for $j = 1, \ldots, m$ but $M_w^{m+1}\varphi \notin L^{1,\infty}.$ However, for $\varphi = Tf, f \in L^1,$ the following holds:

Received by the editors May 20, 1986 and, in revised form, May 31, 1986.
First author supported by grant No. 1593/82 of the Comisión Asesora de Investigación Científica y Técnica, Madrid.
Second author supported by NSF grant DMS-8600699.
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0273-0979/87 $1.00 + .25 per page

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THEOREM. If $T$ is as above, $M_m^w T^* f \in L^{1,\infty}$ for all $f \in L^1$ and all $m \in \mathbb{N}$, and $\|M_m^w T^* f\|_{1,\infty}$ grows as a geometric progression. Hence the same is true for $Tf$.

As in [1], the motivation for this research is the following QUESTION. What is the necessary and sufficient condition on a nonnegative function $\varphi$ for the existence of $f \in L^1$ such that

$$\varphi \leq |Tf| \quad \text{a.e.}$$

In other words, what is the precise description of the magnitude of $Tf$? The theorem gives a necessary condition stronger than $\mathcal{E}L^{1,0}$, namely,

$$\|M_m^w \varphi\|_{1,\infty} \leq C_1 C_m^2$$

but, as shown in the last section of [1], this condition is not sufficient (see §5 of [1] for other remarks concerning this question).

PROOF OF THE THEOREM. Let us first remark that the corresponding result with $Tf$ replaced by the Hardy-Littlewood maximal function $Mf$ is also true. In fact, in this case something more precise is true, namely

$$(3) \quad M^w Mf(x) \leq CMf(x)$$

for some constant $C = C(n)$. This is shown in [1, pp. 9–10], and it is also implicit in [2].

In fact, our proof of the theorem will be a consequence of something similar to (3). We will show that

$$(4) \quad M^w T^* f \leq C\{T^* f + Mf\}.$$ Together with (3) this will give

$$M_m^w T^* f \leq C^m\{T^* f + Mf\}$$

(note that $M_m^w (\varphi + \psi) \leq 2(M_m^w \varphi + M_m^w \psi)$), which clearly implies the theorem.

In order to prove (4), fix $x$ and let $B$ be a ball centered at $x$. Let $2B$ denote the ball having the same center as $x$ and twice the radius and set

$$f_1 = f \chi_{2B}, \quad f_2 = f - f_1.$$ Then $T^* f \leq T^* f_1 + T^* f_2$ and

$$\|T^* f\|_{1,\infty}^B \leq 2(\|T^* f_1\|_{1,\infty}^B + \|T^* f_2\|_{1,\infty}^B).$$

Since $T^*$ satisfies a weak $(1,1)$ estimate, we have

$$(5) \quad \|T^* f_1\|_{1,\infty}^B \leq \|T^* f_1\|_{1,\infty} \leq C\|f_1\|_1 = C \int_{2B} |f(y)| \, dy \leq Cm(B)Mf(x).$$

Now we will prove that for $z \in B$

$$(6) \quad \|T^* f_2(z)\| \leq C(T^* f(x) + Mf(x)).$$

This implies

$$\|T^* f_2\|_{1,\infty}^B \leq Cm(B)(T^* f(x) + Mf(x)),$$
and together with (5) this gives

$$||T^*f||_{1,\infty}^B \leq Cm(B)(T^*f(x) + Mf(x)),$$

which is (4).

For (6) we have to prove that for any $\varepsilon > r$, $r$ being the radius of $B$,

$$|T_{\varepsilon}f_2(z)| \leq C(T^*f(x) + Mf(x)).$$

(7)

Now

$$T_{\varepsilon}f_2(z) = \int_{\mathbb{R}^n \setminus B_{z}} f(y)K(z - y) dy.$$

If $\delta = \varepsilon + r$, it is clear that the contribution in this integral of $B_0 = \delta B/r$, the ball centered at $x$ of radius $\delta$, is dominated (using (1)) by

$$C\varepsilon^{-n} \int_{B_0} |f(y)| dy \leq CMf(x).$$

It remains to estimate

$$I = \int_{|y-x|>\delta} f(y)K(z - y) dy.$$

This is compared with $T_\delta f(x)$ in the usual way:

$$I - T_\delta f(x) = \int_{|y-x|>\delta} f(y)\{K(z - y) - K(x - y)\} dy.$$

Using (2) we obtain

$$|I - T_\delta f(z)| \leq C|z - x| \int_{|y-x|>\delta} |f(y)| \frac{dy}{|y-x|^{n+1}},$$

and it is well known that this is in turn dominated by $Mf(x)$. Therefore we have proved that

$$|T_{\varepsilon}f_2(z)| \leq C(|T_{\delta}f(x)| + Mf(x)),$$

which yields (7) and finishes the proof of the theorem.

REMARK. S. Drury (private communication) has independently generalized some results of [1] proving that $M_wTf \in L^{1,\infty}$, i.e. the case $m = 1$ of the Theorem. He replaces the condition (1) and (2) by the so-called Hörmander condition (see [3, p. 34, condition (2')]).

REFERENCES


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