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Complex manifolds and deformation of complex structures, by Kunihiko Kodaira, with an appendix by Daisuke Fujiwara. Translated by Kazuo Akao. Grundlehren der mathematischen Wissenschaften, vol. 283, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1986, x + 465 pp., $48.00. ISBN 0-387-96188-7

In mathematics and science it is a familiar occurrence to have objects, such as systems of equations, depending on parameters. The investigation of this dependence goes under many names such as the study of bifurcations, or of unfoldings, or of deformations depending on the area. Historically and conceptually, the local deformation theory of compact complex manifolds has played a central role in the modern understanding of these phenomena. *Complex manifolds and deformation of complex structures* is a careful exposition of this local compact complex analytic deformation theory by one of its founders.

Deformation theory is as old as algebraic geometry. Riemann's theory of the theta divisor studied the fine structure of the family of algebraic line bundles on a given algebraic curve with special regard to the relation of the spaces of algebraic sections of the line bundles to the singularities of the theta divisors. (Of course, Riemann used the language of divisors and rational functions; see **[M3]**.) Out of this study of curves came the Riemann-Roch theorem and a count by Riemann of the number of parameters or moduli that an algebraic curve of genus g depends on. In the forties Teichmüller put the notion of moduli of Riemann surfaces on a rigorous basis, using ideas very special to the theory of one complex variable.

In two dimensions there was 19th-century work on families of curves on surfaces and some isolated work of Max Noether on the number of moduli certain algebraic surfaces should depend on. Further invariant theory was concerned with finding the global generic parameters or invariants that various natural families of algebraic geometric objects, such as hypersurfaces in $\mathbf{P}^n$, depend on (see **[Z** and **M2]**).

It is fair to say that at the start of the fifties no general results on deformations of complex or even algebraic manifolds of dimensions two or greater existed. Indeed, simple examples showed new phenomena that did not exist in one dimension, and made it unclear whether the set of such deformations has any structure, local or otherwise. It is worthwhile at this point to mention one of these. There are simple examples of maximal rank algebraic maps, $p\colon X \to \mathbf{P}^1$, from a three-dimensional projective manifold X down to the projective line $\mathbf{P}^1$ such that the fibre over the origin is a nontrivial $\mathbf{P}^1$ bundle over $\mathbf{P}^1$ and the fibre over every point other than the origin is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. This example of a really fine map exhibiting 'popping of the complex structure' at first sight does not seem to bode well for a local theory of continuous variation of complex structures.

In 1957 Frohlicher and Nijenhuis proved the very important result that if the first cohomology of the holomorphic tangent sheaf of a compact complex manifold X is zero, then the manifold X is locally rigid in the sense that, given any holomorphic maximal rank map $p: M \to Y$, where M and Y are complex manifolds, and where an X as above is biholomorphic to $p^{-1}(y)$ for some $y \in Y$, then there is a neighborhood U of y such that $p^{-1}(U) \approx U \times X$. In a famous collaboration, Kunihiko Kodaira and Donald Spencer proceeded to lay down a general local theory of deformations of compact complex manifolds which contained this result as a special case. *Complex manifolds and deformation of complex structures* gives a very readable presentation of these results, one fascinating aspect of which is that Kodaira often stops and explains what he, Spencer, and other researchers knew and were looking for when the foundational results were formulated and proved. The book stops short of proving the theorem of Kuranishi asserting the existence of a local family without any cohomological hypotheses; nonetheless Kuranishi's result is discussed and *Complex manifolds and deformation of complex structures*, with its appendix on elliptic partial differential operators on manifolds by Daisuke Fujiwara, gives all the background needed to study its proof.

Since Kuranishi's result caps the local theory of Kodaira and Spencer and gives a good feeling for a major thrust of this book, it is worth summarizing. Let X be a connected compact complex manifold. There exist complex analytic spaces Y and Z and a proper holomorphic map $p: Y \to Z$ such that:

(a) There is a point $z \in Z$ such that the fibre of p over z is biholomorphic to X.

(b) The map p is flat (which in this context amounts to saying that p is locally a product projection on a neighborhood of $p^{-1}(z)$).

(c) Z is the fibre over the origin of a holomorphic map of a neighborhood U of the origin of $H^1(X, \Theta_X)$ into $H^2(X, \Theta_X)$, where Θ_X is the holomorphic tangent sheaf of X.

(d) Given any other flat map $q: A \to B$ with a fibre $q^{-1}(b)$ biholomorphic to X, there is a holomorphic map f of a neighborhood V of $b \in B$ into Z which induces q from p as a pullback.

The map, f, of V to Z has some uniqueness properties whose exact statement depends on whether $H^0(X, \Theta_X)$ is 0 or not. The tangent space of Z at z is naturally identified with $H^1(X, \Theta_X)$, and the differential df of $T_{B,b} \to H^0(X, \Theta_X)$, which is generally called the Kodaira-Spencer map, can be defined in terms of q and independently of the explicit construction of Z.

The further developments of deformation theory, especially moduli spaces and the global theory of deformations, require the consideration of more general flat holomorphic families over more general base spaces, even if you are only interested in the most classical algebraic varieties (see [**M1**], the appendices to [**M2**] and [**Z**], and [**G**]). For these developments, which grow naturally out of the work of Kodaira and Spencer, the techniques of harmonic theory are not enough. Nonetheless, for those interested mainly in complex manifolds and their differential geometry, *Complex manifolds and deformation of complex structures* is certainly the place to start and may well contain all the information that is wanted.

When I was a graduate student many of us learned harmonic theory on complex manifolds and deformation theory by studying the then-new book, *Complex manifolds*, by Kodaira and J. Morrow [**K + M**]. For the many people who enjoyed studying *Complex manifolds* it is worth noting that *Complex manifolds and deformation structures* is a considerably expanded version of parts of that book.

Certainly those interested in complex geometry (whether algebraic, analytic, or differential geometric) will want a copy of *Complex manifolds and deformation of complex structures* on their bookshelf.

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ANDREW J. SOMMESE

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Semi-Riemannian geometry: *With applications to relativity*, by Barrett O'Neill, Pure and Applied Mathematics, vol. 103, Academic Press, 1983, xiii + 457 pp., $45.00. ISBN 0-12-526740-1

This book gives a thorough introduction to semi-Riemannian (i.e., pseudo-Riemannian) manifolds using the notation of modern differential geometry. The author assumes that the reader has some knowledge of point-set topology, but does not assume a background in differential geometry. Thus, the book begins with a good introduction to smooth manifolds and tensor fields on manifolds.

If M is a smooth manifold, then a semi-Riemannian metric tensor g is a symmetric nondegenerate $(0,2)$ tensor field on M of constant signature. In local coordinates $x_1, x_2, \ldots, x_n$ one writes

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, dx_i\, dx_j.$$

Locally g is represented by the symmetric matrix $(g_{ij} = g_{ij}(x_1, \ldots, x_n))$ of smooth functions of x. This tensor is required to have a constant number s of negative eigenvalues and $n - s$ positive eigenvalues. The index s is zero for Riemannian (i.e., positive definite) manifolds, and for Lorentzian manifolds