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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987

0273-0979/87 $1.00 + $.25 per page

Noncommutative harmonic analysis, by Michael E. Taylor. Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, 1986, xvi + 328 pp., $68.00. ISBN 0-8218-1523-7

Harmonic analysis began as a technique for solving partial differential equations, in the work of Daniel Bernoulli on the vibrating string equation and Fourier on the heat equation. Since then, both subjects have blossomed into independent, wide-ranging, central mathematical disciplines with many subspecialties and with connections to almost all branches of mathematics, pure and applied. I do not think Bernoulli or Fourier would have been surprised by the developments in partial differential equations, but they surely would have been astounded by the growth of harmonic analysis. I also think they would have been pleased by Michael Taylor's new book, which explores some of the recent connections between harmonic analysis and partial differential equations, very much in the spirit of their pioneering work.

The modern definition of harmonic analysis is roughly the following: there is a linear space of functions $\mathscr{F}$, real or complex valued, ordinary or generalized, defined on a domain X on which a group G acts. One seeks first to identify those functions in $\mathscr{F}$ which transform in as simple a fashion as possible under G, then one seeks to expand the general function in $\mathscr{F}$ as a series or integral of these simple functions, and finally one seeks to use the expansion to solve problems which are compatible with the action of G. The simplest cases, such as the Fourier series expansions used by Bernoulli and Fourier, involve abelian groups G. In these cases the simple functions are those which lie in one-dimensional spaces invariant under the group action, such as $f(x) = e^{inx}$ on the circle. But for noncommutative groups there are never enough of these, so the appropriate notion is irreducibility of the span of translates of the function. For compact groups the irreducibility implies that the span of the translates is finite-dimensional, but for noncompact noncommutative groups it is necessary to allow infinite-dimensional spans. Thus we have a hierarchy of complexity: abelian compact $\rightarrow$ abelian noncompact $\rightarrow$ noncommutative compact $\rightarrow$ noncommutative noncompact. The historic development of the subject follows this hierarchy closely, but from our present perspective it is totally misleading. Chapter 1 of Taylor's book concerns the Heisenberg group, which belongs to the most complex category.

Of course the categorization by the type of the group is not as clearcut as it first appears. Consider, for example, the Fourier transform on Euclidean space $\mathbf{R}^n$ (here $X = G =$ additive group of $\mathbf{R}^n$). An abelian group, wouldn't you say? But look again. There is a larger group of Euclidean motions, the semidirect product of the additive $\mathbf{R}^n$ group and the rotation group $\mathrm{SO}(n)$ (or the orthogonal group $\mathrm{O}(n)$ if you allow improper motions). The rotation group is compact but noncommutative, and the motion group is noncommutative and noncompact. These groups play a spirited role in the development of the theory of Euclidean Fourier analysis, as presented in a standard reference work of Stein and Weiss [**S-W**]. In Taylor's book they show up prominently in Chapters 4 and 5. If you like still bigger groups, there is the group of conformal transformations which is lurking in Chapter 10.

There is actually a more refined categorization of Lie groups than the crude one above, given in terms of the Lie algebra. The Lie algebra of a Lie group is just the algebra of invariant first-order differential operators on the group, under the commutator operation, or equivalently the infinitesimal generators of one-parameter subgroups. (All the groups considered in Taylor's book, as indeed all groups related to differential equations, are Lie groups, but there are non-Lie groups that are important in other aspects of harmonic analysis.) The Lie algebra plays the role of a "derivative" of the Lie group. So, for example, the Lie group (if connected) is commutative if and only if the Lie algebra is. Those Lie algebras which are furthest from being commutative are called *semisimple*, as are the associated Lie groups. The technical definition is that the natural bilinear form on the algebra, called the *Killing form*, which measures noncommutativity, be nondegenerate. It is then a theorem that an algebra is semisimple if and only if it is a direct sum of simple (no nontrivial ideals) algebras, which explains the terminology. Among the semisimple Lie algebras, it is easy to recognize those that correspond to compact groups—they are exactly the ones whose Killing form is negative definite. This is a surprising result, since both the compact circle group and the noncompact line have the same commutative Lie algebra. Examples of compact semisimple Lie groups are the rotation and orthogonal groups, the spin groups, and the special unitary groups. Examples of noncompact semisimple groups are the special linear groups (real and complex), the homogeneous Lorentz group, the symplectic and metaplectic groups, and conformal groups.

At the other extreme, the noncommutative Lie algebras that are closest to being commutative are the *nilpotent* ones, the definition being that all sufficiently long products vanish. Examples of nilpotent Lie groups are upper triangular matrices with ones on the diagonal, and the Heisenberg group. Again there is a structure theorem: the connected and simply connected nilpotent Lie groups are just Euclidean spaces with product defined by polynomial laws. A slightly more noncommutative class of Lie algebras are the *solvable* ones, defined by requiring iterated squaring to kill the Lie algebra. The solvable Lie groups that appear in Taylor's book are all semidirect products, such as the Euclidean motion group, the Poincaré (or inhomogeneous Lorentz) group, and the two-dimensional $ax + b$ group. It is really the semidirect product structure ($G = HN$ where N is a normal abelian subgroup) rather than the solvability that plays a crucial role in the discussion.

This completes the cast of characters, as far as groups are concerned. Why does the Heisenberg group get star billing? There are a number of compelling reasons. We recall the definition, first of the Lie algebra: h_n is the $(2n + 1)$-dimensional algebra generated by the operators $\partial/\partial x_1, \dots, \partial/\partial x_n$, multiplication by $x_1, \dots, x_n$, and the identity. The only nontrivial commutation relations are Heisenberg's canonical commutation relations $[\partial/\partial x_j, x_j] = I$. The associated simply connected group is easily described by the multiplication rule

$$(t_1, q_1, p_1) \circ (t_2, q_2, p_2) = \left(\left(t_1 + t_2 + \tfrac{1}{2}(p_1 \cdot q_2 - p_2 \cdot q_1\right), q_1 + q_2, p_1 + p_2\right)$$

for $t_j \in \mathbf{R}$, $q_j, p_j \in \mathbf{R}^n$. The representation theory of the Heisenberg group is especially simple: there is only one interesting representation, and we have already described it, at least on the Lie algebra level, by our description of h_n. Actually one obtains a whole family of representation from this one by composing with the dilations

$$\delta_{\pm\lambda}(t, p, q) = \left(\pm\lambda t, \pm\lambda^{1/2} q, \lambda^{1/2} p\right).$$

Aside from some unimportant one-dimensional representations, these are all the irreducible representations (this is the Stone–von Neumann theorem). It is easy to get a Fourier inversion formula and Plancherel formula using these representations from the analogous results for the line. But now the basic representation of the Heisenberg group begins to pay dividends. Since the Lie algebra generators are represented by differentiation and coordinate multiplication, the alert reader will suspect pseudodifferential operators are waiting in the wings. And indeed they are; both the Kohn-Nirenberg calculus and the Weyl calculus emerge naturally when the basic representation is exponentiated. This approach explains the connection between the two and shows clearly why the Weyl calculus is more natural.

The Heisenberg group begets the symplectic groups $Sp(n, \mathbf{R})$ of linear transformations preserving the skew-symmetric form

$$\sigma((q_1, p_1), (q_2, p_2)) = p_1 \cdot q_2 - q_1 \cdot p_2$$

in that these give rise to automorphisms $(t, q, p) \to (t, \sigma(q, p))$. The basic representation of h_n extends in a natural way to the Lie algebra of the symplectic group, and this exponentiates to the famous metaplectic representation, not of the symplectic group itself but of its two-fold covering group.

The Heisenberg group has a basic subelliptic differential operator, the Heisenberg Laplacian. When viewed through the basic representation, this operator becomes the harmonic oscillator $-\Delta + |x|^2$, whose spectral theory leads to the Hermite functions. All these aspects of the Heisenberg group are featured prominently in Taylor's book (they are also discussed in an article by Roger Howe [**H**] in this Bulletin). There are still other reasons for liking the Heisenberg group: for example, in several complex variables, it plays the role of a model space for the boundary of a strictly pseudoconvex domain in the work of Folland and Stein [**F-S**]. In fact, choose any current research journal and you are likely to find an article about some aspect of the Heisenberg group.

Partial differential equations comprise the other half of the story in Taylor's book. What kind of equations? Basically the big three equations of mathematical physics: Laplace equation, heat equation, and wave equation, only modified so the basic Laplacian part reflects the noncommutative groups under discussion. For example, on the Heisenberg group it is

$$\sum_{j=1}^{n} \left(\left(\frac{\partial}{\partial q_j} - \frac{1}{2} p_j \frac{\partial}{\partial t} \right)^2 + \left(\frac{\partial}{\partial p_j} + \frac{1}{2} q_j \frac{\partial}{\partial t} \right)^2 \right).$$

If we write Δ for any of these general Laplacians on X, then the heat and wave operators on $\mathbf{R} \times X$ are $\partial/\partial t - \Delta$ and $\partial^2/\partial t^2 - \Delta$, respectively. One can also consider a Laplace operator $\partial^2/\partial t^2 + \Delta$ on $\mathbf{R} \times X$. There are a number of elementary connections among these operators. For example, one can pass back and forth between the Laplace and wave equations by analytic continuation in the t variable from t to it (similarly, from the heat equation one can obtain the Schrödinger equation). And the Laplace equation can be solved using the heat equation and the subordination principle. On a formal level the solutions of all these equations can be given in terms of functions of the Laplacian Δ on X, as $e^{t\Delta}$, $\cos t\sqrt{-\Lambda}$, $e^{-t\sqrt{-\Lambda}}$, for heat, wave, and Laplace equation, respectively. On the other hand, once one understands the solution $\cos t\sqrt{-\Delta}$ of the wave equation, one can construct a general calculus of functions of Δ by Fourier synthesis,

$$f(\sqrt{-\Lambda}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) \cos t\sqrt{-\Lambda}\, dt$$

for suitable even functions f. There are other ways as well to construct such a functional calculus, and Taylor's book is brimming over with explicit computations in a variety of contexts.

Indeed, this book is a cornucopia of formulas. Scarcely a page goes by without at least a half-dozen numbered displays. Mostly these are identities, such as the Plancherel formula, or the Poisson integral. One aspect of contemporary harmonic analysis that is absent from this book is the L^p estimate: for this the reader will have to consult other works (such as Stein-Weiss [**S-W**], Stein [**S**], and the more recent Garcia-Cuerva–Rubio de Francia [**G-R**]).

This is a book for mature readers. The author does not hesitate to use ideas and results from diverse branches of mathematics—special functions, functional analysis, partial differential equations, differential geometry. But for the reader with a strong background, or a willingness to accept a non-self-contained presentation, this book offers many pleasures. In addition to the concrete computational results already mentioned, the book contains concise and insightful presentations of a number of important abstract topics, including induced representations, representations of compact groups, conformal transformations, Clifford algebras and spinors. Taylor has an original point of view, and is able to bring new insights to familiar topics.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 1, July 1987

0273-0979/87 $1.00 + $.25 per page

Near-rings and their links with groups, by J. D. P. Meldrum, Pitman Publishing, Boston, London, Melbourne, 275 pp., $44.95. ISBN 0-470-20648-9.

Many mathematicians do not highly appreciate theories which have prefixes like near-, semi-, hemi-, para-, quasi-, and so on. This certainly should not apply in the case of near-rings. As the name suggests, a *near-ring* is a "generalized ring"; more precisely, the commutativity of addition is not required and just one of the distributive laws is postulated.

Near-rings arise very naturally in the study of mappings on groups. If $(G, +)$ is a group (not necessarily abelian) then the set $M(G)$ of all mappings from G to G is a near-ring with respect to pointwise addition and composition of mappings. If G is abelian and if one only takes "linear" maps