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THE ARNOL'D FORMULA FOR ALGEBRAICALLY COMPLETELY INTEGRABLE SYSTEMS

J.-P. FRANÇOISE

Let $F: V^n \to \mathbf{R}^m$ be a *real algebraic mapping* and let us denote by $D \subset \mathbf{R}^m$ the set of its critical values. We assume that $F: V^n \setminus F^{-1}(D) \to \mathbf{R}^m \setminus D$ is a proper topological fibration so that we can consider the real monodromy of F defined as the action of $\pi_1(\mathbf{R}^m \setminus D)$ on $H_*(F^{-1}(c), \mathbf{Z})$, $c \in \mathbf{R}^m \setminus D$. We propose to study the real monodromy of mappings F which are defined on a symplectic manifold (V^{2m}, ω) and whose generic fibers are Lagrangian for the symplectic form ω. Such mappings are momentum mappings of integrable Hamiltonian systems. This particular case is interesting because we know well the topology of the fibers [**Fo**] and because the real monodromy relates to the monodromy of the actions of the integrable system [**D**] via the Arnol'd formula [**AR**]. In that case, the connected components of the fibers are tori. When F is associated to a symplectic action of a torus, the fibers are connected [**At**]; in some cases they may be not connected (as in example (c)).

If $c_0 \in \mathbf{R}^m \setminus D$, there is a neighborhood $U = F^{-1}(T)$, $c_0 \in T \subset \mathbf{R}^m \setminus D$ which retracts by deformation on the fiber $F^{-1}(c_0)$. On U, there is a 1-form η [**GS**] such that $\omega_{|U} = d\eta$. Let $\gamma_j(c)$, $j = 1, \ldots, m$, $c \in T$, be a set of generators of $H_1(F^{-1}(c), \mathbf{Z})$, $c \in T$; we define locally on U the actions p_j by the *Arnol'd formula* [**Ar**]:

$$p_j = \int_{\gamma_j(c)} \eta.$$

To the symplectic form $\omega_{|U} = \sum_{i=1}^m dF_i \wedge \eta_i$ is associated the period matrix $\psi_{ij} = \int_{\gamma_j(c)} \eta_i$ and the Stokes formula gives

$$dp_j/dF_i = F^* \psi_{ij} \qquad [\mathbf{H\ddot{o}}].$$

Two types of obstructions to the global existence of actions have to be carefully distinguished. If $H_2(V^{2m}, \mathbf{R}) \neq 0$, the cohomology class of ω is an obvious obstruction of topological nature [**D**]. If ω is exact, the nonexistence of global actions can be expressed precisely by the multivaluedness of the Arnol'd integrals. It is then an obstruction of analytical nature.

For *algebraically completely integrable systems* (we refer to [**AM**] for a complete definition), one can introduce another monodromy. We will denote again the complexified mapping $F: V_{\mathbf{C}}^{2m} \to \mathbf{C}^m$ and $D_{\mathbf{C}} \hookrightarrow \mathbf{C}^m$ its set of critical values. Let us assume that there is a family of curves $\mathcal{C}_c$ ($c \in \mathbf{C}^m$) generically smooth of genus g such that its discriminant Δ contains $D_{\mathbf{C}}$ and such

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that the Hamiltonian flows X_j of the F_j induce linear flows on the Jacobian of $\mathcal{C}_c$. Given an a.c.i. system, there might be several associated curves perhaps of different genuses (for instance, the Lagrange top [**RM**]; see also [**AM**, **Ha**]). According to Picard-Fuchs theory as exposed in [**BK**], associated to the bundle $\mathcal{C}$ with fiber $\mathcal{C}_c$ over $\mathbf{C}^m \setminus \Delta$, there is a monodromy that we call the monodromy of the curve $\mathcal{C}$. For hyperelliptic families, it has been studied in [**Ac**].

THEOREM. *If $g \geq m$, for any $c \in \mathbf{C}^m \setminus \Delta$, the value of ψ_{ij} at c is a sum of Abelian integrals of the first kind on $\mathcal{C}_c$.*

COROLLARY. *To compute the actions, one has to determine the generators $\gamma_j(c)$ of the real tori and to integrate the period matrix ψ_{ij}.*

It might be quite complicated to find the real generators in the complex tori. See [**NV**] for a general "algebro-topological" program. Following these steps, one gets new derivations of the preceding results for the action-angles of the Toda Lattice [**FM**, **M**], Neumann system [**Mo**], and a new computation for the Kowalevskaya top [**Fr**].

EXAMPLES. (a) *The Euler top*, when complexified, is a.c.i. with nontrivial curve monodromy, but it has no monodromy of the actions. The set of critical values of the reduced system (F_1, F_2) is the union of three half-lines ($F_1 = AF_2$, $F_1 = BF_2$, $F_1 = CF_2$, $F_2 \geq 0$; A, B, C are the diagonal components of the tensor of inertia). Thus $\pi_1(\mathbf{R}^2 \setminus D)$ is zero and there is no real monodromy.

(b) *The spherical pendulum* was studied in [**D** and **C**]. The set of critical values D contains an isolated point, so there is a possibility of real monodromy, and in fact Duistermaat proved that the actions have monodromy. The system is a.c.i. for the family of elliptic curves $z^2 = \Phi_c(w) = 2(F_1 - w) \cdot (1 - w^2) - F_2$. Since $g < m$, one cannot apply the theorem, but a direct computation gives the actions as integrals over paths in the curve of abelian forms of the *three* possible kinds.

(c) *The Kowalevskaya top*. The set of critical values D is of codimension 1, so there is no real monodromy and no monodromy of the actions. The topology of the fibers studied by [**K**] changes following bifurcations described recently in a quite general set-up by Fomenko [**Fo**]. The number of connected components of the regular fiber is 1, 2, or 4. The monodromy of the curve is not trivial. The Theorem gives a method to compute the actions as integrals over paths on the curve $\mathcal{C}_c$. Even so there is no monodromy of the actions; they are not globally defined, as shows the explicit expression obtained [**Fr**].

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UNIVERSITÉ PARIS-SUD, MATHÉMATIQUES, BÂT. 425, F-91405 ORSAY CEDEX, FRANCE