In this note, we describe an extension of Hilbert’s Theorem 90 for $K_2$ of fields to the relative $K_2$ of semilocal PID’s containing a field. Most of the results for $K_2$ of fields proven in [M-S and S] then carry over to the relative $K_2$ of semilocal PID’s containing a field, e.g. computation of the torsion subgroup, and the norm residue isomorphism. Applying this to the semilocal ring of $\{0, 1\}$ in $A^1_E$, for a field $E$, gives a computation of the torsion and co-torsion in

$$K_3(E)\text{ind} := K_3(E)/K_3^M(E).$$

Specifically, we have

1. The torsion subgroup of $K_3(E)\text{ind}$ is $H^0_\text{ét}(E, \mu_2 \otimes \mathbb{Q}_l)$.  
2. $K_3(F, \mathbb{Z}/n)^\text{ind} \simeq H^1_\text{ét}(E, \mu_2 \otimes \mathbb{Z}_l(2))$ for $(n, \text{char}(E)) = 1$, so

$$\lim_{n} K_3(E)^\text{ind} / l^n \rightarrow H^1_\text{ét}(E, \mathbb{Z}_l(2))$$

for $l$ prime, $l \neq \text{char}(E)$.

3. $K_3(E)^\text{ind}$ satisfies Galois descent for extensions of degree prime to $\text{char}(E)$.

4. Bloch’s group $B(E)$ is uniquely $n$-divisible if $E$ contains an algebraically closed field, and $(n, \text{char}(E)) = 1$.

The results (3) and (4) follow directly from (1) and (2). To prove (1) and (2), the essential case is when $E$ is a finite extension of the prime field; when $E$ has positive characteristic (1) and (2) follow from Quillen’s computation of the $K$-theory of finite fields [Q2]. For $E$ a number field, (1) and (2) are the conjectures of Lichtenbaum and Quillen in the case of $K_3$, i.e. if $E$ is a number field, the Chern class

$$c_{2,1} : K_3(E)\text{ind} \otimes \mathbb{Z}_l \rightarrow H^1_\text{ét}(E, \mathbb{Z}_l(2))$$

is an isomorphism. Merkurjev and Suslin have obtained these results, using similar methods. Here we give a sketch of the proof of Hilbert’s Theorem 90 for relative $K_2$, and its application to the Lichtenbaum-Quillen conjecture for $K_3$.

Let $R$ be a semilocal PID with Jacobson radical $I$. Let $\mathcal{D}$ be an Azumaya algebra over $R$, and $X$ the associated Brauer-Severi scheme over $R$. Let $\overline{X}$ denote the fiber over $\overline{R} := R/I$. There is an $E_1$ spectral sequence converging to the relative $K$-theory $K_*(X, \overline{X})$ analogous to the Quillen spectral sequence converging to $K_*(X)$; the $E_2$ term $E_2^{p,q}(X, \overline{X})$ is a relative analogue
LEMMA 1. Assume that $\mathcal{D}$ is split, and $\mathcal{D}$ has prime rank $l$ over $R$. Let $h : R \to R'$ be a finite étale extension. Then

$$E_2^{1,-2}(X, \overline{X}) \to E_2^{1,-2}(X_{R'}, \overline{X}_{R'})$$

is injective.

Let $S$ be a semilocal PID with Jacobson radical $J$. Let $L$ be the quotient field of $S$. Weibel [W] has shown that $K_2(S, J)$ is generated by symbols $\{a, b\}$, with $a \in (1 + J)^*$, $b \in L^*$. Suppose that $S$ contains a field $K$ containing a prime. Let $\sigma$ be a generator of $\text{Gal}(S^\alpha/S)$, let

$$N : K^*(S^\alpha, J^\alpha) \to K^*(S, J)$$

be the norm map, and let $\sigma$ be a generator of $\text{Gal}(S^\alpha/S)$.

LEMMA 2. $\{y, 1 - N(y)\}$ is in $(1 - \sigma)K_2(S^\alpha, J^\alpha)$ for all $y \in (1 + J^\alpha)^*$.

PROOF (SKETCH). In [S], this is done by an easy direct computation. We proceed here by a generic element method.

Let $F_0$ be the prime field. If $F_0 = \mathbb{Q}$, let $R = \mathbb{Q}(\zeta_4)[t]/(t)$; if $F_0 = \mathbb{F}_p$, let $R = \mathbb{F}_p(\zeta_4, t_0)[t]/(t)$, with $t_0$ and $t$ indeterminants. If $E$ is an extension ring of $F_0$ let $R_E = E[t]/(t)$. We let $k_0$ be the ground field $\mathbb{Q}(\zeta_4)$ or $\mathbb{F}_p(\zeta_4, t_0)$. If $T$ is an $R$-scheme let $T^\text{an}$ denote the fiber over $\overline{R} := R/(t)$. After making a purely transcendental extension, we may assume that $S$ contains $k_0$.

Let $x_0, \ldots, x_{l-1}, v$ be indeterminants over $k$, and let $u = v^l$ if $l \neq p$; if $l = p$, let $u = v^p - v$. Let $A^0$, $A$, and $B$ be the rings

$$A^0 = k_0[x_0, \ldots, x_{l-1}], \quad A = A^0[u], \quad B = A^0[v],$$

so $B = A[v]$. Let $x$ be the element

$$x = 1 + t \sum x_i v^i \in R_B,$$

so $x$ is the “generic element” of the universal Kummer extension (or Artin-Schreier extension if $l = p$) $R_B/R_A$ with $x \equiv 1 \mod t$.

Let $N : R_B \to R_A$ be the norm, $\sigma$ the generator of $\text{Gal}(R_B/R_A)$ with $\sigma(v) = \zeta_4 v$ for $l \neq p$, $\sigma(v) = v + 1$ for $l = p$. Let $X^{1/l} = \text{Spec}(R_B)$, $X = \text{Spec}(R_A)$ and $X^0 = \text{Spec}(R_{A^0})$. Let $W$ be the closed subscheme of $X^{1/l}$ defined by the ideal $((1 - N(x))/t)$, $W'$ the subscheme defined by $(x)$.

The symbol $\{x, 1 - N(x)\}$ defines an element of $K_2(X^{1/l} - (W \cup W'))$, $X^{1/l} - \overline{W}$).

There is an affine open subset $U$ of $X^{1/l} - W - W'$, containing the generic point of $X^{1/l}$, and an element $\mu$ of $K_2(U, \overline{U})$ with

$$\{x, 1 - N(x)\} = \mu^\sigma/\mu \quad \text{in } K_2(U, \overline{U}).$$

This is fairly easy to show, the essential points being

1. the inclusion $X^{1/l} \to X^{1/l}$ is split,
2. $X^{1/l}$ and $X$ are both affine lines over $X^0$.
Given an element \( y \) of \( S^\alpha \), if \( y \) is sufficiently general, we can pull back the relation \((*)\) to show that \( \{ y, 1 - N(y) \} = z^\sigma / z \) in \( K_2(S^\alpha, J^\alpha) \). We then conclude by a specialization argument. \( \square \)

Let \( \{ S_i \mid i \in I \} \) be a filtering direct system of semilocal PID’s. Let \( J_i \) be the Jacobson radical of \( S_i \), and let \( S_\infty \) and \( J_\infty \) denote the direct limits

\[
S_\infty = \lim \rightarrow S_i, \quad J_\infty = \lim \rightarrow J_i.
\]

We suppose that \( \{ S_i \mid i \in I \} \) satisfies

(I) Every \( x \) in \( 1 + J_\infty \) is a norm from \( S_\infty \).

(II) If \( P(u) \) is a separable polynomial with coefficients in \( S_\infty \) and has degree \( d < l \), then \( P(u) \) factors completely in \( S_\infty \).

**Lemma 3.** Assuming (I) and (II), the quotient group

\[
K_2(S_\infty^\alpha, J_\infty^\alpha)/(1 - \sigma)K_2(S_\infty^\alpha, J_\infty^\alpha)
\]

is generated via symbols by \((1 + J_\infty)^* \otimes L_\infty^*\).

The proof is essentially the same as the proof of the similar fact for \( K_2 \) of fields in [B–T].

**Theorem 1 (Hilbert’s Theorem 90 for relative \( K_2 \)).** Let \( S \) be a semilocal PID containing a field \( k \), and containing an \( l \)th root of unity, \( l \) a prime. Let \( J \) be the Jacobson radical of \( S \) and \( \alpha \) a unit in \( S \). Let \( \sigma \) be a generator of \( \text{Gal}(S^\alpha/S) \). Then the complex

\[
K_2(S^\alpha, J^\alpha) \to (1 - \sigma)K_2(S^\alpha, J^\alpha) \to K_2(S, J)
\]

is exact.

Using the above lemmas, the proof follows the same outline as Suslin’s proof of Hilbert’s Theorem 90 for \( K_2 \) of fields in [S].

Exactly as in Suslin [S], applying Hilbert’s Theorem 90 to the generic Kummer extension \( S(u^{1/l})/S(u) \), and the generic Artin-Schreier extension \( S(P^{-1}(u))/S(u) \) one gets

**Theorem 2.** Let \((S, J)\) be as above. The \( l \)-torsion subgroup of \( K_2(S, J) \) is generated by symbols \( \{ f, \zeta \} \), where \( f \) is in \( (1 + J)^* \), and \( \zeta \) is an \( l \)th root of unity. \( K_2(S, J) \) is \( p \)-torsion free if \( k \) has characteristic \( p > 0 \).

**Corollary.** Let \( E \) be a field. Then the \( l \)-torsion subgroup of \( K_3(E)^{\text{ind}} := K_3(E)/K_3^M(E) \) is cyclic.

**Proof.** We may assume that \( E \) contains \( \mu_l \). Let \((R, J)\) be the semilocal ring of \( \{0, 1\} \) on \( A_E^1 \). We have the exact sequence

\[
0 \to K_3(E)^{\text{ind}} \to K_2(R, J) \to K_2(R) \to .
\]

From this and Theorem 2, it follows that \( iK_3(E)^{\text{ind}} \) is generated by symbols of the form \( \{ f, \zeta \}, f \in (1 + J)^* \) with \( f \in (R^*)^l \). Writing such an \( f \) as \( f = g^l \), \( g \in R^* \), we normalize \( g \) so that \( g(0) = 1 \). Then the class of \( f \mod((1 + J)^*)^l \) is determined by the value \( g(1) \in \mu_l \), proving the corollary. \( \square \)
Now we can show

**Theorem 3.** Let $E$ be a number field. The Chern class

$$c_{2,1}: K_3(E)^{\text{ind}} \otimes \mathbb{Z}_l \to H^1(E, \mathbb{Z}_l(2))$$

is an isomorphism, so the $l$-primary torsion in $K_3(E)^{\text{ind}}$ is isomorphic to $H^0(E, (\mu_l^\infty)^{\otimes 2})$.

**Proof.** We may assume that $E$ contains $\mu_l$. From [Q], $K_3(E)$ is finitely generated. From the above, the $l$-torsion in $K_3(E)^{\text{ind}}$ is cyclic, hence the $l$-primary torsion is also cyclic. By [B-T], $K_3^M(E)$ is a 2-torsion group; by [B] the rank of $K_3(E)$ is $r_2$. Thus $K_3(E)^{\text{ind}}/l$ is a $\mathbb{Z}/l$ vector space of dimension between $r_2$ and $1 + r_2$. In addition, the Chern class vanishes on the Milnor $K_3$ (this follows from the integral product formula for Chern classes). Let $symb: H^1(E, \mathbb{Z}/l^2)^{-} \to K_2(E)$ be the map

$$c_{2,1} \circ K_3(E, \mathbb{Z}/l^2) \to H^1(E, \mathbb{Z}/l^2)^{-} \otimes \mu_l \to iK_2(E)$$

and let $H$ be the kernel of $symb$. Tate [T] has shown that $H$ is $(\mathbb{Z}/l)^{1+r_2}$ and that $symb$ is surjective. Soule [So] has shown that $c_{2,1}$ is surjective. Suslin shows in [S] that $H = c_{2,1}(K_3(E))$, and that the induced map

$$c_{2,1}: iK_2(E) \to H^1(E, \mu_l^{\otimes 2})/H$$

is inverse to $symb$. This, together with the computation of $K_3(E)^{\text{ind}}/l$ above, implies that the Chern class map

\[ c_{2,1}: K_3(E, \mathbb{Z}/l)^{\text{ind}} \to H^1(E, \mu_l^{\otimes 2}) \]

is surjective. Let $R$ be as in the corollary, $\pi^*: E \to R$ the inclusion. A localization argument together with (*) shows that

\[ c_{2,1}: K_3(R; \mathbb{Z}/l)^{\text{ind}} \to H^1(R, \mu_l^{\otimes 2}) \]

is surjective. We have the commutative square

\[
\begin{array}{ccc}
K_3(R; \mathbb{Z}/l) & \overset{\delta_K}{\longrightarrow} & K_3(E; \mathbb{Z}/l) \\
c_{2,1} \downarrow & & \downarrow c_{2,1} \\
H^1(R, \mu_l^{\otimes 2}) & \overset{\delta_H}{\longrightarrow} & H^1(E, \mu_l^{\otimes 2})
\end{array}
\]

where the $\delta$'s are the maps “reduce mod $J$” followed by the difference map. This diagram, together with the surjectivity of $c_{2,1}$ and $\delta_H$, then implies that $\delta_K$ is surjective (since it is obviously surjective on the Milnor $K_3$), and hence $K_2(R, J; \mathbb{Z}/l) \to K_2(R; \mathbb{Z}/l)$ is injective. Thus $K_2(R, J)/l \to K_2(R)/l$ is injective.

Let $L$ be the quotient field of $R$ and $i: \text{Spec}(L) \to \text{Spec}(R)$ the inclusion. We have the commutative ladder

\[
\begin{array}{cccc}
K_3(E)^{\text{ind}}/l^n & \to & K_2(R, J)/l^n & \to & K_2(R)/l^n & \to & (K_2(E)/l^n)^2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
H^1(E, \mu_l^{\otimes 2}) & \to & H^2(R, i_!(\mu_l^{\otimes 2})) & \to & H^2(R, \mu_l^{\otimes 2}) & \to & (H^2(E, \mu_l^{\otimes 2}))^2 & \to & 0
\end{array}
\]

the horizontal lines coming from the relativization sequence, and the vertical arrows Chern classes (Galois symbols). For all $n$, the Galois symbols for
$K_2(R)/l^n$ and $K_2(E)/l^n$ are isomorphisms. The surjectivity of $\delta_H$ shows that $H^2(R, i_!(\mu_l^{\otimes 2})) \to H^1(R, \mu_l^{\otimes 2})$ is injective, hence the second vertical arrow is an isomorphism for $n = 1$. We have the commutative ladder

$$
\begin{array}{cccccc}
\text{symb} & K_2(R, J) & K_2(R, J)/l^n & K_2(R, J)/l^{n+1} & K_2(R, J)/l & 0 \\
H^1(R, i_!(\mu_l^{\otimes 2})) & \downarrow & \downarrow & \downarrow & \downarrow & \\
H^2(R, i_!(\mu_l^{\otimes 2})) & \to & H^2(R, i_!(\mu_l^{\otimes 2}_{l+1})) & \to & H^2(R, i_!(\mu_l^{\otimes 2})) \\
\end{array}
$$

with the second row exact, and the first row exact except possibly at $K_2(R, J)/l^n$. This and induction show that the Galois symbol for $K_2(R, J)/l^n$ is an isomorphism for all $n$.

From the localization sequence on $A^1_E$, together with a knowledge of $K_2(E)$, and $K_1$ of number fields, it follows that $K_2(R, J)$ has no $l$-divisible subgroups, hence the same for $K_2(R, J)$. Thus for $n$ sufficiently large, the $l$-primary torsion in $K_3(E)^{\text{ind}}$ injects into $K_2(R, J)/l^n$. From the ladder (**), it follows that the Chern class $c_{2,1} : K_3(E)^{\text{ind}} \to H^1(E, \mu_l^{\otimes 2})$ is injective on the $l$-primary torsion for large $n$. From this, the surjectivity of $c_{2,1}$, and the computation of the ranks of $K_3(E)^{\text{ind}}$ and $H^1(E, Z_l(2))$ (the latter due to Tate [T]) it follows that the Chern class gives an isomorphism on the limits

$$c_{2,1} : K_3(E)^{\text{ind}} \otimes Z_l \to H^1(E, Z_l(2)),$$

proving the theorem. 

REFERENCES


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