GENERALIZED ALBANESE VARIETIES
FOR SURFACES

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In this paper we announce a solution to the generalized Albanese problem for smooth projective surfaces. More precisely, for such a surface $X$ over a field $k$ and for each modulus $m$ (see next paragraph) we show the existence of a pair $(G, \alpha)$, where $G$ is a commutative algebraic group over $k$ (or more generally a principal homogeneous space under such a group), $\alpha: X \to G$ is a rational map, and any rational map with modulus $m$ factors through $\alpha$.

Let $X$ be such a surface and let $U = X \sim \bigcup D_j$ be the complement of a finite number of integral divisors on $X$. In [2, Chapter 3, Proposition 1] it was shown that for a rational map $\alpha: X \to G$ into an algebraic group we get a homomorphism $\gamma_m: C_m(X) \to G(k)$ for some modulus $m$, where $C_m(X)$ denotes the $K$-theoretic idele class group of $X$. When $\text{domain}(\alpha) = U$ we have $m = \sum m_j D_j$ with $m_j \geq 1$. In this situation we say that $\alpha$ admits $m$ as modulus.

It is clear that by usual descent arguments we may assume that $k$ is algebraically closed and work with algebraic groups rather than principal homogeneous spaces.

Let $\text{Cat}_m$ denote the category of maps $\alpha: X \to G$ which admit $m$ as modulus.

**THEOREM 1.** In $\text{Cat}_m$ there exists $\alpha: X \to G_{um}$ with the universal mapping property described above.

**SKETCH OF THE PROOF.** By [5, Corollary to Theorem 2] it suffices to show that the dimension of algebraic groups $G$ with $\beta: X \to G$ in $\text{Cat}_m$ and $\beta$ maximal [5, Definition 2] is bounded. For this by blowing up points in $U$ we reduce to the case of a Lefschetz pencil $\pi: X' \to \mathbb{P}^1$ with $m$ flat over $\mathbb{P}^1$.

Then by using [2, Chapter 3, Lemma 1] we see that $(\beta, \pi): X'' \to G \times S$ admits $m$ as a modulus in the sense of [6, Definition 1] ($X'' \to S \subset \mathbb{P}^1$ is the smooth part of the pencil). Hence it factors through the relative generalized jacobian $J_m$ of $X''$ [6, Theorem 1]. Then it is easy to see that the dimension of the group generated by $\beta$ is equal to the dimension of the image of the composite map

$$J_m \to G \times S \xrightarrow{\text{proj}} G.$$ 

Therefore if $\beta$ generates $G$ then $\dim(G) \leq \dim(J_m)$.

**REMARK.** We can give an alternate proof of Theorem 1 by applying [7, §3, Proposition 4] to show that $\alpha$ admits $m$ as modulus iff

$$\alpha^*(\Omega_G^{\text{inv}}) \subset (H^0(U, \Omega_U)^d=0 \cap H^0(X, \Omega_X(-m))).$$

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This ties Theorem 1 with the modulus defined by Faltings and Wüstholz [1, Theorem 1] in characteristic zero.

For the construction of the pair $(G_{um}, \alpha)$ we have

THEOREM 2. In characteristic zero, the universal pair can be constructed by rigidifying the Picard functor $\text{Pic}_{\text{Pic}_X^0}$ of the Picard variety $\text{Pic}_X^0$ of $X$.

SKETCH OF THE CONSTRUCTION. We know that $G_{um}$ must be an extension of the Albanese variety $\text{Alb}_X$ of $X$ by a connected algebraic group. Therefore it comes from a rigidification of $\text{Pic}_{\text{Pic}_X^0}$ [3] (for rigidification see [4, Definition 2.1.1]). The rigidifier $R$ is supported on $\{x_1, \ldots, x_r, 0\}$ in $\text{Pic}_X^0$, where $x_1, \ldots, x_r$ is a set of free generators for the image of

$$\text{Kernel}(ZD_1 + \cdots + ZD_n \rightarrow \text{Pic}_X(k) \rightarrow \text{Pic}_X(k)/\text{Pic}_X^0(k))$$

and 0 is the zero section. For a given $m$ we can determine $R$ explicitly (for a special case see [3]). Then $\alpha$ is obtained simply by using the definition of the rigidified Picard functor.

REMARKS. (1) For $m' \geq m$ we have an affine morphism $G_{um'} \rightarrow G_{um}$, hence $\lim G_{um}$ exists. This pro-smooth group is important for the class-field theory of $X$.

(2) The homomorphism $\gamma_m: C_m(X) \rightarrow G_{um}(k)$ is surjective because for $x$ in $U$, 1 in $C_m(x)$ is mapped to $\alpha(x)$ and $\alpha$ generates $G_{um}$. In characteristic zero it seems natural to expect that when we restrict to the idele classes of degree zero, $\gamma_m$ is an isomorphism iff $p_g(X) = 0$.

The details together with the discussion of the relative case and the extension to dimensions $> 2$ will appear elsewhere.

REFERENCES


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