DYSON’S CRANK OF A PARTITION

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1. Introduction. In [3], F. J. Dyson defined the rank of a partition as the largest part minus the number of parts. He let \( N(m, t, n) \) denote the number of partitions of \( n \) of rank congruent to \( m \) modulo \( t \), and he conjectured

\[
N(m, 5, 5n + 4) = \frac{1}{5}p(5n + 4), \quad 0 \leq m \leq 4;
\]
\[
N(m, 7, 7n + 5) = \frac{1}{7}p(7n + 5), \quad 0 \leq m \leq 6,
\]

where \( p(n) \) is the total number of partitions of \( n \) [1, Chapter 1]. These conjectures were subsequently proved by Atkin and Swinnerton-Dyer [2].

Dyson [3] went on to observe that the rank did not separate the partition of \( 11n + 6 \) into 11 equal classes even though Ramanujan’s congruence

\[
p(11n + 6) \equiv 0 \pmod{11}
\]

holds. He was thus led to conjecture the existence of some other partition statistic (which he called the crank); this unknown statistic should provide a combinatorial interpretation of \( \frac{1}{11}p(11n + 6) \) in the same way that (1.1) and (1.2) treat the primes 5 and 7.

In [4, 5], one of us was able to find a crank relative to vector partitions as follows:

For a partition \( \pi \), let \( \#(\pi) \) be the number of parts of \( \pi \) and \( \sigma(\pi) \) be the sum of the parts of \( \pi \) (or the number \( \pi \) is partitioning) with the convention \( \#(\emptyset) = 0 \) for the empty partition \( \emptyset \). Let

\[
V = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into distinct parts,}
\]
\[
\pi_2, \pi_3 \text{ are unrestricted partitions}\}.
\]

We shall call the elements of \( V \) vector partitions. For \( \vec{\pi} = (\pi_1, \pi_2, \pi_3) \) in \( V \) we define the sum of parts, \( s \), a weight, \( \omega \), and a crank, \( r \), by

\[
s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3),
\]
\[
\omega(\vec{\pi}) = (-1)^{\#(\pi_1)},
\]
\[
r(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).
\]

We say \( \vec{\pi} \) is a vector partition of \( n \) if \( s(\vec{\pi}) = n \). For example, if

\[
\vec{\pi} = (5 + 3 + 2, 2 + 2 + 1, 2 + 1 + 1)
\]
then \( s(\bar{\pi}) = 19 \), \( \omega(\bar{\pi}) = -1 \), \( r(\bar{\pi}) = 0 \) and \( \bar{\pi} \) is a vector partition of 19. The number of vector partitions of \( n \) (counted according to the weight \( \omega \)) with crank \( m \) is denoted by \( N_v(m, n) \), so that

\[
N_v(m, n) = \sum_{\bar{\pi} \in V \atop s(\bar{\pi}) = n \atop r(\bar{\pi}) = m} \omega(\bar{\pi}).
\]

The number of vector partitions of \( n \) (counted according to the weight \( \omega \)) with crank congruent to \( k \) modulo \( t \) is denoted by \( N_v(k, t, n) \), so that

\[
N_v(k, t, n) = \sum_{m=-\infty}^{\infty} N_v(mt + k, n) = \sum_{\bar{\pi} \in V \atop r(\bar{\pi}) \equiv k \pmod{t}} \omega(\bar{\pi}).
\]

By considering the transformation that interchanges \( \pi_2 \) and \( \pi_3 \) we have

\[
N_v(m, n) = N_v(-m, n)
\]

so that

\[
N_v(t - m, t, n) = N_v(m, t, n).
\]

We have the following generating function for \( N_v(m, n) \):

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - zq^n)(1 - z^{-1}q^n)}.
\]

By putting \( z = 1 \) in (1.11) we find

\[
\sum_{m=-\infty}^{\infty} N_v(m, n) = p(n).
\]

**Vector-Crank Theorem (Garvan [4, 5]).**

\[
N_v(0, 5, 5n + 4) = N_v(1, 5, 5n + 4) = \cdots = N_v(4, 5, 5n + 4) = \frac{p(5n + 4)}{5},
\]

\[
N_v(0, 7, 7n + 5) = N_v(1, 7, 7n + 5) = \cdots = N_v(6, 7, 7n + 5) = \frac{p(7n + 5)}{7},
\]

\[
N_v(0, 11, 11n + 6) = \cdots = N_v(10, 11, 11n + 6) = \frac{p(11n + 6)}{11}.
\]

The above still leaves open the question of whether there is a crank for ordinary partitions. The answer is “yes” when the crank is defined as follows:

**Definition.** For a partition \( \pi \), let \( l(\pi) \) denote the largest part of \( \pi \), \( \omega(\pi) \) denote the number of ones in \( \pi \), and \( \mu(\pi) \) denote the number of parts of \( \pi \) larger than \( \omega(\pi) \). The crank \( c(\pi) \) is given by

\[
c(\pi) = \begin{cases} 
    l(\pi) & \text{if } \omega(\pi) = 0, \\
    \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0.
\end{cases}
\]
Our main result is the following.

**Theorem 1.** The number of partitions \( \pi \) of \( n \) with \( c(\pi) = m \) is \( N_V(m,n) \) for all \( n > 1 \).

Obviously, in light of the Vector-Crank Theorem, we see that Theorem 1 supplies the crank asked for by Dyson.

2. **Proof of Theorem 1.** We shall require the standard notation of \( q \)-series:

\[
(A; q)_n = (A)_n = \prod_{j=0}^{\infty} \frac{(1 - Aq^j)}{(1 - Aq^{j+n})}
\]

\((= (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}) \) when \( n \) is a positive integer), and

\[
(A; q)_\infty = (A)_\infty = \prod_{j=0}^{\infty} (1 - Aq^j).
\]

We now transform (1.11):

\[
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n)z^mq^n = \frac{(1 - q)}{(zq)_\infty} \frac{(q^2; q)_\infty}{(q/z)_\infty}
\]

\[= \frac{(1 - q)}{(zq)_\infty} \sum_{j=0}^{\infty} \frac{(zq)_j(q/z)^j}{(q)_j} \quad \text{(by [1, p. 17])}
\]

\[= \frac{(1 - q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^jz^{-j}}{(q^2; q)_{j-1}(zq^j+1)_\infty}.
\]

As was noted in (1.12), when we set \( z = 1 \) the series on the left of (2.3) reduces to the generating function for \( p(n) \). For \( j > 0 \), the \( j \)th term in the sum on the right is

\[
z^{-j}q^{j+1 \cdots +1} \frac{(1 - q^2)(1 - q^3) \cdots (1 - q^j)(1 - zq^{j+1})(1 - zq^{j+2}) \cdots}
\]

The standard techniques of partition theory [1, Chapter 1] show that this expression generates partitions with \( \omega(\pi) = j \) and the exponent on \( z \) is clearly \( \mu(\pi) - \omega(\pi) \), i.e. \( c(\pi) \), since \( j > 0 \).

Thus we must interpret

\[
\frac{(1 - q)}{(1 - zq)(1 - zq^2)(1 - zq^3) \cdots}
\]

as the generating function for partitions without ones. By considering conjugate partitions, we note that

\[
\frac{1}{(1 - zq)(1 - zq^2)(1 - zq^3) \cdots}
\]
generates all partitions with the exponent on $z$ counting the largest part, and for integers larger than 1

\[ \frac{q}{(1 - q)(1 - q^2)(1 - q^3) \cdots} \]

generates partitions with at least one 1 appearing again with the exponent on $z$ counting the largest part. Note that this interpretation fails for 1 because this is the unique instance in which introducing a 1 into the partitions of $n - 1$ alters the largest part. Hence

\[ \frac{1 - q}{(zq)_{\infty}} \]

counts (for $n > 1$) the number of partitions with no ones and with the exponent on $z$ being the largest part of the partition $l(\pi) = c(\pi)$. Thus in the double series expansion of

\[ \frac{(1 - q)}{(zq)_{\infty}} + \sum_{j=1}^{\infty} \frac{q^2 z^{-j}}{(q^2; q)_{j-1}(zq^j+1)_{\infty}} \]

we see that the coefficient of $z^m q^n$ ($n > 1$) is the number of ordinary partitions of $n$ in which $c(\pi) = m$. Therefore by (2.3), we have Theorem 1. □

3. Conclusion. We can’t resist exhibiting $c(\pi)$ for the first instance of (1.3).

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<th>$\mu(\pi)$</th>
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As Theorem 1 together with (1.15) predicts, $c(\pi)$ provides eleven different residue classes modulo 11.

References


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