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DYSON'S CRANK OF A PARTITION

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1. Introduction. In [**3**], F. J. Dyson defined the rank of a partition as the largest part minus the number of parts. He let $N(m,t,n)$ denote the number of partitions of n of rank congruent to m modulo t, and he conjectured

$$N(m,5,5n+4) = \tfrac{1}{5}p(5n+4), \qquad 0 \le m \le 4; \tag{1.1}$$

$$N(m,7,7n+5) = \tfrac{1}{7}p(7n+5), \qquad 0 \le m \le 6, \tag{1.2}$$

where $p(n)$ is the total number of partitions of n [**1**, Chapter 1]. These conjectures were subsequently proved by Atkin and Swinnerton-Dyer [**2**].

Dyson [**3**] went on to observe that the rank did not separate the partition of $11n+6$ into 11 equal classes even though Ramanujan's congruence

$$p(11n+6) \equiv 0 \pmod{11} \tag{1.3}$$

holds. He was thus led to conjecture the existence of some other partition statistic (which he called the crank); this unknown statistic should provide a combinatorial interpretation of $\frac{1}{11}p(11n+6)$ in the same way that (1.1) and (1.2) treat the primes 5 and 7.

In [**4**, **5**], one of us was able to find a crank relative to vector partitions as follows:

For a partition π, let $\#(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π (or the number π is partitioning) with the convention $\#(\phi) = \sigma(\phi) = 0$ for the empty partition ϕ, of 0. Let

$$V = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into distinct parts, } \pi_2, \pi_3 \text{ are unrestricted partitions}\}.$$

We shall call the elements of V *vector partitions*. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in V we define the sum of parts, s, a weight, ω, and a crank, r, by

$$s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3), \tag{1.4}$$

$$\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}, \tag{1.5}$$

$$r(\vec{\pi}) = \#(\pi_2) - \#(\pi_3). \tag{1.6}$$

We say $\vec{\pi}$ is a vector partition of n if $s(\vec{\pi}) = n$. For example, if

$$\vec{\pi} = (5+3+2, 2+2+1, 2+1+1)$$

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then $s(\vec{\pi}) = 19$, $\omega(\vec{\pi}) = -1$, $r(\vec{\pi}) = 0$ and $\vec{\pi}$ is a vector partition of 19. The number of vector partitions of n (counted according to the weight ω) with crank m is denoted by $N_V(m,n)$, so that

$$N_V(m,n) = \sum_{\substack{\vec{\pi}\in V\\ s(\vec{\pi})=n\\ r(\vec{\pi})=m}} \omega(\vec{\pi}). \tag{1.7}$$

The number of vector partitions of n (counted according to the weight ω) with crank congruent to k modulo t is denoted by $N_V(k,t,n)$, so that

$$N_V(k,t,n) = \sum_{m=-\infty}^{\infty} N_V(mt+k,n) = \sum_{\substack{\vec{\pi}\in V\\ s(\vec{\pi})=n\\ r(\vec{\pi})\equiv k \pmod t}} \omega(\vec{\pi}). \tag{1.8}$$

By considering the transformation that interchanges π_2 and π_3 we have

$$N_V(m,n) = N_V(-m,n) \tag{1.9}$$

so that

$$N_V(t-m,t,n) = N_V(m,t,n). \tag{1.10}$$

We have the following generating function for $N_V(m,n)$:

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \tag{1.11}$$

By putting $z=1$ in (1.11) we find

$$\sum_{m=-\infty}^{\infty} N_V(m,n) = p(n). \tag{1.12}$$

VECTOR-CRANK THEOREM (GARVAN [**4**, **5**]).

(1.13)

$$N_V(0,5,5n+4) = N_V(1,5,5n+4) = \cdots = N_V(4,5,5n+4) = \frac{p(5n+4)}{5},$$

(1.14)

$$N_V(0,7,7n+5) = N_V(1,7,7n+5) = \cdots = N_V(6,7,7n+5) = \frac{p(7n+5)}{7},$$

$$N_V(0,11,11n+6) = \cdots = N_V(10,11,11n+6) = \frac{p(11n+6)}{11}. \tag{1.15}$$

The above still leaves open the question of whether there is a crank for ordinary partitions. The answer is "yes" when the crank is defined as follows:

DEFINITION. For a partition π, let $l(\pi)$ denote the largest part of π, $\omega(\pi)$ denote the number of ones in π, and $\mu(\pi)$ denote the number of parts of π larger than $\omega(\pi)$. The crank $c(\pi)$ is given by

$$c(\pi) = \begin{cases} l(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

Our main result is the following.

THEOREM 1. *The number of partitions π of n with $c(\pi) = m$ is $N_V(m,n)$ for all $n > 1$.*

Obviously, in light of the Vector-Crank Theorem, we see that Theorem 1 supplies the crank asked for by Dyson.

2. PROOF OF THEOREM 1. We shall require the standard notation of q-series:

$$(A;q)_n = (A)_n = \prod_{j=0}^{\infty} \frac{(1-Aq^j)}{(1-Aq^{j+n})} \tag{2.1}$$

($= (1-A)(1-Aq)\cdots(1-Aq^{n-1})$ when n is a positive integer), and

$$(A;q)_\infty = (A)_\infty = \prod_{j=0}^{\infty}(1-Aq^j). \tag{2.2}$$

We now transform (1.11):

$$\begin{aligned} \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_V(m,n) z^m q^n &= \frac{(1-q)}{(zq)_\infty}\cdot\frac{(q^2;q)_\infty}{(q/z)_\infty} \\ &= \frac{(1-q)}{(zq)_\infty}\sum_{j=0}^{\infty}\frac{(zq)_j(q/z)^j}{(q)_j} \quad \text{(by [1, p. 17])} \\ &= \frac{(1-q)}{(zq)_\infty} + \sum_{j=1}^{\infty}\frac{q^j z^{-j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty}. \end{aligned} \tag{2.3}$$

As was noted in (1.12), when we set $z = 1$ the series on the left of (2.3) reduces to the generating function for $p(n)$. For $j > 0$, the jth term in the sum on the right is

$$\frac{z^{-j}q^{\overbrace{1+1+\cdots+1}^{j \text{ times}}}}{(1-q^2)(1-q^3)\cdots(1-q^j)(1-zq^{j+1})(1-zq^{j+2})\cdots}.$$

The standard techniques of partition theory [1, Chapter 1] show that this expression generates partitions with $\omega(\pi) = j$ and the exponent on z is clearly $\mu(\pi) - \omega(\pi)$, i.e. $c(\pi)$, since $j > 0$.

Thus we must interpret

$$\frac{(1-q)}{(1-zq)(1-zq^2)(1-zq^3)\cdots}$$

as the generating function for partitions without ones. By considering conjugate partitions, we note that

$$\frac{1}{(1-zq)(1-zq^2)(1-zq^3)\cdots}$$

generates all partitions with the exponent on z counting the largest part, and for integers larger than 1

$$\frac{q}{(1-zq)(1-zq^2)(1-zq^3)\cdots}$$

generates partitions with at least one 1 appearing again with the exponent on z counting the largest part. Note that this interpretation fails for 1 because this is the unique instance in which introducing a 1 into the partitions of $n-1$ alters the largest part. Hence

$$\frac{1-q}{(zq)_\infty}$$

counts (for $n > 1$) the number of partitions with no ones and with the exponent on z being the largest part of the partition $l(\pi) = c(\pi)$. Thus in the double series expansion of

$$\frac{(1-q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty}$$

we see that the coefficient of $z^m q^n$ $(n > 1)$ is the number of ordinary partitions of n in which $c(\pi) = m$. Therefore by (2.3), we have Theorem 1. □

3. Conclusion. We can't resist exhibiting $c(\pi)$ for the first instance of (1.3).

partitions of 6	$l(\pi)$	$\omega(\pi)$	$\mu(\pi)$	$c(\pi)$
6	6	0	1	6
$5+1$	5	1	1	0
$4+2$	4	0	2	4
$4+1+1$	4	2	1	−1
$3+3$	3	0	2	3
$3+2+1$	3	1	2	1
$3+1+1+1$	3	3	0	−3
$2+2+2$	2	0	3	2
$2+2+1+1$	2	2	0	−2
$2+1+1+1+1$	2	4	0	−4
$1+1+1+1+1+1$	1	6	0	−6

As Theorem 1 together with (1.15) predicts, $c(\pi)$ provides eleven different residue classes modulo 11.

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