28. W. Schwarz, *Remarks on the theorem of Elliott and Daboussi; and applications*, Elementary and Analytic Theory of Numbers (H. Iwaniec, ed.), Banach Center Publications, Inst. of Math. Polish Acad. Sci., Warszawa, 1985, pp. 463–498.

29. W. Schwarz and J. Spilker, *Eine Anwendung des Approximationssatzes von Weierstrass-Stone auf Ramanujan-Summen*, Arch. Wisk. (3) **19** (1971), 198–209.

30. F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–355.

31. E. C. Titchmarsh, *The Riemann zeta function*, Oxford, 1951; new edition (D. R. Heath-Brown, ed.), Oxford Univ. Press, 1986.

32. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen. II*, Acta Math. Acad. Sci. Hungar. **18** (1967), 411–467.

P. D. T. A. ELLIOTT

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Le problème des modules pour les branches planes, by Oscar Zariski, with an appendix by Bernard Teissier. Hermann, Paris, 1986, 209 pp., $28.00. ISBN 2-7065-6036-4

In the fall of 1973 Oscar Zariski gave a series of lectures about curve singularities at the École Polytechnique in Paris. A set of notes based on these lectures was prepared by François Kmety and Michel Merle, and an appendix was added by Bernard Teissier. These notes have now been published as a book by Hermann.

Nothing of comparable originality has been published about the subject since the work of Enriques and Chisini [**5**]. The book describes a deep and beautiful analogy between the moduli space $\mathcal{M}_g$ for smooth curves of genus g, and a certain local moduli space $\mathcal{M}_\Gamma$ for plane curve singularities. A partial description of $\mathcal{M}_\Gamma$ is given, and many important open problems are described.

Riemann noticed that smooth curves of genus g depend on $3g-3$ parameters if $g>1$. How many parameters are needed to describe plane curve singularities of the same topological type? It is remarkable that we are still unable to solve this problem in general, and in this book the reader will find the first real progress toward a solution.

Chapter I, "Préliminaires", Chapter II, "Invariants d'équisingularité", and Chapter III, "Représentations paramétriques", give a clear review of the classical theory of plane curve singularities. Chapter IV, "L'espace des modules", and Chapter V, "Étude des exemples..." give detailed calculations. Chapter VI, "Le point de vue de la théorie des déformations", and Teissier's appendix, describe the modern theory in which deformation theory plays a central role.

The vanishing of a polynomial $f(X,Y)\in \mathbf{C}[X,Y]$ defines an affine plane curve. A singularity of this curve, for example at the origin, is described as follows. As an element of the power series ring $\mathbf{C}[[X,Y]]$, $f(X,Y)$ will factor into a finite number of irreducible power series, with multiplicities. An irreducible factor $g(X,Y)$ defines a *branch* C of the singularity, with coordinate ring $\mathcal{O}=\mathbf{C}[[X,Y]]/(g)=\mathbf{C}[[t^n,y(t)]]$. In other words the problem of studying the singularity can be reduced to the problem of studying complete local rings of this type. These "*branches planes*" C, sometimes identified with their coordinate rings $\mathcal{O}$, are the objects studied in this book.

The power series $y(t)$ is called a *Puiseaux series*; the valuation v on the integral closure $\overline{\mathcal{O}} = \mathbf{C}[[t]]$ restricted to $\mathcal{O}$ gives a semigroup $\Gamma = v(\mathcal{O}) \subset N$ called the *semigroup of the branch*; Γ can be described in terms of a finite set of pairs of integers $(m_1, n_1), \ldots, (m_g, n_g)$ called the *characteristic pairs*.

In the late twenties it was observed [**1**, **2**, **12**] that these concepts have topological meaning. The intersection of a small 3-sphere centered at the origin with the curve $f = 0$ is a *link*: a finite union of circles, one for each branch. The components of the link are *local knots*, and these can be described in terms of the characteristic pairs [**4**, **8**]. The genus of the local knot (the genus of its Seifert surface) is the number of characteristic pairs g. So it is reasonable to regard the set of branches with the same set of characteristic pairs, or with the same semigroup Γ, as having the same topological type, and with this topological type there is an associated notion of genus g.

In this book the set of branches with the same topological type is called an *equisingularity class*. Zariski developed his theory of equisingularity in [**13**, **14**, and **15**]. These important papers form a bridge between the classical theory of branches [**5**, **10**], with its emphasis on infinitely near points, and the modern theory.

By defining and studying the *saturation* of a local ring [**15**], Zariski put the classical results in a new light. The equisingular families of [**14**] are related to the equisingular deformations which play a central role in this book. It was while he was working on equisingularity that Zariski became interested in the moduli problem.

In Chapter I the notion of equisingularity class is defined, as in [**13**], in terms of infinitely near points. Two branches are said to be *equisingular* if their multiplicity sequences (the sequences of multiplicities of successive quadratic transforms) are the same. Equisingularity is clearly an equivalence relation. If C is a branch, the set of branches equisingular with C is called $L(C)$. The moduli space $\mathcal{M}(C) = L(C)/\simeq$ is defined to be the set $L(C)$ modulo the equivalence relation of analytic isomorphism (isomorphism of the complete local rings $\mathcal{O}$ above).

It is then stressed that $\mathcal{M}(C)$, given a suitable topology, will rarely be separated. This is because there are special branches, just as there are special curves (curves with automorphisms), and the moduli space $\mathcal{M}_g$ can only be shown to be an algebraic variety if it is defined as general curves (curves without automorphisms), modulo isomorphism. Define a branch $\mathcal{O}$ to be *general* if the dimension of the module of deformations $T^1(\mathcal{O}/\mathbf{C}, \mathcal{O})$ is minimal among branches with the same semigroup, and define $\mathcal{M}_\Gamma$ to be the set of general branches with semigroup Γ, modulo analytic isomorphism.

$\mathcal{M}_\Gamma$ is a dense open subset of $\mathcal{M}(C)$, and the largest open subset on which one can hope to put an algebraic structure. The book calls $\mathcal{M}_\Gamma$ the *generic component* of $\mathcal{M}(C)$: it is the exact local analog of $\mathcal{M}_g$.

Now we can state the central problem: is $\mathcal{M}_\Gamma$ an algebraic variety? If so, what are its properties? In particular, what is its dimension?

To study this question it is clearly convenient to have several different definitions of equisingularity. Chapter II is concerned with proving equivalences between these definitions, and with showing that the *conductor* c is an invariant of the equisingularity class.

Associated with a branch $\mathcal{O}$ we have the semigroup Γ; the characteristic pairs; the multiplicity sequence $e_1, e_2, \ldots$; the *characteristic* $(n, \beta_1, \ldots \beta_g)$ (certain integers associated with the Puiseaux expansion); a minimal set of generators for Γ, $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$; and the local knot type. Fixing any of these data determines the equisingular type. Most of this is proved here (Theorems 3.3 and 3.9), and the reader can trace the remaining equivalences in the references. The assertion about the conductor is established by proving the classical formula $c = \sum_i e_i(e_i - 1)$. The proof given here is based on a relation between the conductor and the discriminant which Zariski adapted for the purpose from the book of Hecke [**6**].

The material in Chapter II depends on the results in Zariski's papers on equisingularity, but also complements them in a useful way. The chapter ends with further material relating the conductor and properties of differential forms. Here, and again at the end of the next chapter, the reader will see hints of deformation theory.

In Chapter III the parametric representation of branches is studied in more detail. After an automorphism of $\mathcal{O}$ the Puiseaux series $y(t)$ can be replaced by a polynomial in t with highest-degree term $a_{c-1}t^{c-1}$. Thus $\mathcal{M}_\Gamma$, whatever it may be, is finite-dimensional in some sense. A representation of a branch by a Puiseaux series of this kind is called *short*. The chapter ends with some material from Zariski's paper [**16**]. The Theorem in that paper reappears, generalized and in a deformation-theoretic guise, as the "Théorème du saut de τ" [**11**], and on p. 208 of the book under review.

Now we can see the real difficulty. Among all branches isomorphic to $\mathcal{O}$, which has the simplest Puiseaux series? Any isomorphism between two branches is given by a change in the parameter t, so this is a question about the action of the group $\mathrm{Aut}(\mathbf{C}[[t]])$ or, after the reduction above, of the group $\mathrm{Aut}(\mathbf{C}[[t]]/(t^c))$. This algebraic group is far from being reductive (it is equal to its radical), so the usual technique of applying geometric invariant theory does not work. If $g = 1$, however, the problem can be stated very directly.

In the case of a single characteristic pair (m, n) the dimension $D = D(m, n)$ of $\mathcal{M}_\Gamma$ is the smallest integer such that the Puiseaux series of any general branch with this characteristic pair can be put in the form $y(t) = t^m + a_1 t^{v_1} + \cdots + a_{D+1} t^{v_{D+1}}$ for increasing integers $m < v_1 < \cdots < v_{D+1}$. What is $D(m, n)$? The problem is serious: examples suggested (and later results proved) that in this case about half the terms of a short representation could be eliminated by isomorphisms. How many?

This problem was of particular interest to Zariski. He filled several notebooks with calculations relating to the moduli problem, mostly in the case $g = 1$. He obtained explicit formulas for $D(m, n)$ in a number of cases, and believed that there was an explicit formula for $D(m, n)$ in general.

Chapters IV and V consist mainly of calculations from Zariski's notebooks. They show the details of the problem of computing $\mathcal{M}(C)$ with great clarity.

These calculations and others were used in attempts to find a formula for $D(m,n)$.

In an important paper [**3**], Charles Delorme gave a formula for $D(m,n)$ in general. After this, it was noticed that the same formula could be derived using deformation theory (the formula is a little complicated, so we will not give it here). Thus there is a natural and explicit formula for $D(m,n)$: this confirmed Zariski's intuition.

The homothety $t \mapsto \lambda t$ gives a $\mathbf{C}^*$-action on the coefficients $(a_1, \dots a_{D+1})$ of the Puiseaux series. This is equivalent to the action of a finite group on the projective space $\mathbf{P}^D$ (the quotient is a weighted projective space). An unpublished lemma of Allan Adler implies that the quotient, obviously unirational, is in fact rational. Thus we have the

THEOREM. *If $g = 1$, $\mathcal{M}_\Gamma$ is an open subset of a rational variety of dimension $D(m,n)$.*

Little else is known about $\mathcal{M}_\Gamma$. The theorem is incomplete in one respect: we do not have an explicit description of general branches in the case $g = 1$. In every case given here they occur on the complement of the locus of vanishing of a certain discriminant. Describing the general branches explicitly would complete the description of $\mathcal{M}_\Gamma$ in the case $g = 1$: this is perhaps the most accessible unsolved problem in the theory.

Chapter VI, about 50 pages long, treats the problem of obtaining a formula for the dimension of $\mathcal{M}_\Gamma$ in the case of a single characteristic pair (m,n). The results are Zariski's. A notion of equisingular deformation is introduced, and a formula for the dimension is given (Theorem 3.1) which asserts, in effect, that the dimension is the rank of the module of equisingular deformations of the versal equisingular deformation of the special branch $\mathbf{C}[[t^n, t^m]]$. An explicit formula for the dimension is given in the case $(m,n) = (n+1,n)$ (Theorem 4.16), and an explicit formula is stated, without proof, in the case $(m,n) = (hn+1,n)$, for $h > 1$. These results preceded, and inspired, the more general theorem above.

The techniques used here are more general than they seem. For example there is a connection, not well understood, between rings with no equisingular deformations (singularities with modulus number zero) and rings of finite Cohen-Macaulay type.

The remainder of the book consists of Teissier's appendix, also about 50 pages long, divided into two Chapters and an Annexe. Teissier works in the category of analytic spaces. He uses some results proved in joint work with Monique Lejeune-Jalabert [**7**]. There is also a connection with the work of Henry Pinkham [**9**].

The Appendix alone is worth the price of the book. The Annexe gives a treatment of the deformation theory of an isolated complete intersection—one of the few places in the literature where this important topic is discussed. The two Chapters apply it to the moduli problem: let C^Γ be the singularity whose coordinate ring is the semigroup ring of Γ. If Γ is the semigroup of a plane branch, C^Γ is a complete intersection (Proposition 2.2). (Ring theorists will find the Appendix interesting.)

If $\mathcal{O}$ is a plane branch the powers of the maximal ideal in the integral closure $\overline{\mathcal{M}}^i$ induce a filtration on $\mathcal{O}$. The associated graded ring, $\overline{\mathrm{gr}}_{\mathcal{M}}\mathcal{O}$, is isomorphic to the semigroup ring of the semigroup of $\mathcal{O}$ (Proposition 1.2.3). Since a commutative ring is a flat deformation of any associated graded ring, any branch with semigroup Γ is a flat deformation of C^Γ (Theorem 1).

Teissier then constructs a mini-versal semigroup constant deformation of C^Γ (Theorem 3). Every branch with semigroup Γ is a fiber of this family. After some technical material, a formula for the dimension of $\mathcal{M}_\Gamma$ is given (Theorem 6). (Like the Theorem in Chapter VI, this asserts in effect that the dimension is the rank of the module of equisingular deformations of the versal semigroup constant deformation of C^Γ; the formula is not explicit.)

Like the global moduli problem, the local problem is different if $g > 1$. The reader who wishes to study it will find many hints here about how to proceed. The key to the problem is understanding how the parameters defining the versal semigroup constant deformation of C^Γ behave under restriction to a general fiber. The $\overline{\mathrm{gr}}$-construction breaks up the defining equation of a general branch into pieces, which define C^Γ; restriction to a general fiber reverses the process.

The problems which have been solved for $\mathcal{M}_g$ are open for $\mathcal{M}_\Gamma$. Is $\mathcal{M}_\Gamma$ always irreducible? What is its dimension? When is it rational? Does it become a variety of general type if g is large?

Could this theory have an application to physics? In superstring theory, physicists study functions on $\mathcal{M}_g$. Could functions on $\mathcal{M}_\Gamma$ be of interest as well? Blaine Lawson has observed that there is no intrinsic reason to prefer nonsingular curves to singular ones in this theory. If physicists considered the singular case, they could deepen their theory in two ways.

First of all, links and local knots (they are "strings") would become available as models for fundamental particles. Then, one could attempt to give the subtle counts of local parameters in Zariski's theory an interpretation. Perhaps the dimension of the moduli space $D(m,n)$ in the case $g = 1$ is the number of parameters necessary to describe toroidal paths of particles, of homotopy type (m,n), in a force field of some kind.

This book is remarkable for the richness of its results and techniques; for the many problems, solved and unsolved, which it describes; and for the historical sweep of its point of view. But most of all it looks to the future, as Zariski did until the end of his life: one sees here an elegant and powerful theory in its infancy, which one day will be a central part of the theory of curves.

REFERENCES

1. K. Brauner, *Zur Geometrie der Funktionen zweier complexen Veränderlichen*, III, IV, Abh. Math. Sem. Hamburg **6** (1928), 8–54.

2. W. Burau, *Kennzeichnung der Schlauchknoten*, Abh. Math. Sem. Hamburg **9** (1932), 125–133.

3. C. Delorme, *Sur la dimension d'un espace de singularité*, C. R. Acad. Sci. Paris **280** (1975), 1287–89.

4. D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Studies, No. 110, Princeton, 1985.

5. F. Enriques and O. Chisini, *Lezioni sulla theorie geometrica delle equazioni e delle funzioni algebriche*, 3 vols., Bologna, 1915, 1918, and 1925.

6. E. Hecke, *Lectures on the theory of algebraic numbers*, Graduate Texts in Math., vol. 77, Springer-Verlag, Berlin and New York, 1981.

7. M. Lejeune and B. Teissier, *Clôture integrale des ideaux et équisingularité*, Sém. École Polytechnique, (1973–1974).

8. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, No. 61, Princeton, 1968.

9. H. C. Pinkham, *Deformations of algebraic varieties with G_m-action*, Astérisque **20** (1974).

10. J. G. Semple and G. T. Kneebone, *Algebraic curves*, Oxford, 1959.

11. S. Washburn, *Analytically trivial families of local rings*, J. Algebra **35** (1975), 528–533.

12. O. Zariski, *On the topology of algebroid singularities*, Amer. J. Math. **54** (1932), 453–465.

13. ——, *Studies in equisingularity*. I. *Equivalent singularities of plane algebroid curves*, Amer. J. Math. **87** (1965), 507–536.

14. ——, *Studies in equisingularity*. II. *Equisingularity in codimension* 1 (*and characteristic zero*), Amer. J. Math. **87** (1965), 972–1006.

15. ——, *Studies in equisingularity*. III. *Saturation of local rings and equisingularity*, Amer. J. Math. **90** (1968), 961–1023.

16. ——, *Characterization of plane algebroid curves whose module of differentials has maximum torsion*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 781–786.

SHERWOOD WASHBURN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Partial differential relations, by Mikhael Gromov. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris and Tokyo, 1986, ix + 363 pp., $60.00. ISBN 0-387-12177-3

In this important book, Gromov studies very general classes of partial differential equations and inequalities, many of which arise from problems in differential geometry. Using a variety of surprising and intricate techniques, he shows that in many cases these partial differential relations satisfy the "h-principle", i.e., they admit rich families of solutions whenever the appropriate topological obstructions vanish.

Most of the ideas presented here have their origins in a series of papers which Gromov wrote in Russian in the later 60s and early 70s, some alone and some in collaboration with Eliashberg and Rochlin. Thanks to the excellent lecture notes of Haefliger [**H**] and Poenaru [**P**], the earliest part of this work is reasonably well-known. However, this is just the tip of the iceberg: the later papers contain many more, totally original ideas. Unfortunately, these papers were sketchily written, and contained various references to other papers which never appeared. Gromov has devoted a great deal of effort over the past few years to working out these ideas. The end result is this magnificent book.

The core of the book is a series of abstract and powerful theorems. These include a sharp version of the Nash-Moser implicit function theorem which is specific to partial differential operators, as well as much more geometric results such as the main flexibility theorem and the theorems about convex