THE NUMBER OF EQUATIONS NEEDED TO DEFINE
AN ALGEBRAIC SET

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Let $B$ be a commutative Noetherian ring, $X = \text{Spec } B$ the associated affine scheme, $I \subset B$ an ideal and $V = V(I) \subset X$ the closed subset defined by $I$.

**DEFINITION.** Elements $f_1, \ldots, f_s \in I$ define $V$ set-theoretically (equivalently, $V$ is defined set-theoretically by $s$ equations $f_1 = 0, f_2 = 0, \ldots, f_s = 0$) if $\sqrt{(f_1, \ldots, f_s)} = \sqrt{I}$.

Hilbert's Nullstellensatz implies that in the case when $B$ is a finitely generated algebra over an algebraically closed field $k$, this definition agrees with the usual one, i.e., all $f_1, \ldots, f_s$ vanish at a $k$-rational point if and only if it belongs to $V$. In the sequel “defined” always means “defined set-theoretically”.

The question we are dealing with here concerns the minimum number of equations needed to define a given $V \subset X$. A classical result that goes back to L. Kronecker [Kr] says that if $B$ is $n$-dimensional, then $n + 1$ equations would suffice for every $V \subset X$. Our first theorem describes those $V \subset X$ which can be defined by $n$ equations.

**THEOREM A.** Let $k$ be an algebraically closed field, $X$ a smooth affine $n$-dimensional variety over $k$ with coordinate ring $B$, and $V = V' \cup P_1 \cup P_2 \cup \cdots \cup P_r$ an algebraic subset of $X = \text{Spec } B$, where $V'$ is the union of irreducible components of positive dimensions and $P_1, P_2, \ldots, P_r$ some isolated closed points (which do not belong to $V'$). Then $V$ can be defined by $n$ equations if and only if one of the following conditions holds.

(i) $r = 0$, i.e., $V$ consists only of irreducible components of positive dimension.

(ii) $V'$ is empty, i.e., $V$ consists only of closed points and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1 P_1 + n_2 P_2 + \cdots + n_r P_r = 0$ in $A_0(X)$.

(iii) $V'$ is nonempty, $r \geq 1$ and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1 P_1 + n_2 P_2 + \cdots + n_r P_r$ belongs to the image of the natural map $A_0(V') \to A_0(X)$ induced by the inclusion $V' \to X$.

Here $A_0(\cdot)$ stands for the group of zero-cycles modulo rational equivalence [Fu].

**SKETCH OF PROOF.** Our proof consists of three steps. In Step 1 we construct an ideal $I \subset B$ such that $\sqrt{I}$ is the defining ideal of $V$, and in addition $I$ has some other special properties. In Step 2 we, in a special way, pick some ideals $Q_1, \ldots, Q_n$ such that $\sqrt{Q_i}$ is a maximal ideal containing $I$ for each $i$ and $J/J^2$ is $n$-generated, where $J = I \cap Q_1 \cap \cdots \cap Q_n$. In Step 3 we
prove that the 0-dimensional Segre class of $J$ equals 0, so by [Mu2, Theorem 2] and the Suslin cancellation theorem [Su], $J$ is $n$-generated. □

If $B$ is not assumed to be regular, the result of Theorem A need not hold. In fact, for every $n \geq 0$, we have constructed an example of a finitely generated $n$-dimensional algebra $B$ over a suitable algebraically closed field, such that its singular locus consists of just one closed point and for every $d$ between 1 and $n - 1$ it contains a $d$-dimensional subvariety which cannot be defined by $n$ equations.

Storch [St] and Eisenbud-Evans [EE] proved that every algebraic set in $\mathbb{A}^n_k$ can be defined by $n$ equations, while Cowsik-Nori [CN] proved that every curve in $\mathbb{A}^n_k$, where $\text{char} k = p > 0$, can be defined by $n - 1$ equations. Our next theorem sharpens these results.

**THEOREM B.** If $\text{char} k = p > 0$, then every algebraic set $V \subset \mathbb{A}^n_k$ consisting only of irreducible components of positive dimensions can be defined by $n - 1$ equations.

**SKETCH OF PROOF.** The main idea of the proof is contained in the following lemma.

**LEMMA.** Let $f_1, \ldots, f_r, g_1, \ldots, g_r \in I$ and $a \in B$ such that

(i) $f_1, \ldots, f_r$ generates $I_a$ up to radical.
(ii) $g_1, \ldots, g_r$ generate $(I + (a))/(a) \subset B/(a)$ up to radical. Then there exist $r + 1$ elements $h_1, \ldots, h_{r+1}$ which generate $I$ up to radical.

Now assume all irreducible components of $V$ have dimension $d \geq 1$ and use induction on $d$, the case $d = 1$ being settled by [CN]. Let $I = I(V) \subset B = k[x_1, \ldots, x_n]$ be the defining ideal of $V$. By a change of variables we can assume that $k[x_n]$ has zero intersection with every minimal prime over-ideal of $I$. By induction the extension of $I$ in $k(x_n)[x_1, \ldots, x_{n-1}]$ can be generated up to radical by $n - 2$ elements. Thus there exists a square-free polynomial $a(x_n)$ such that $I_{a(x_n)}$ can be generated up to radical by $n - 2$ elements. Since $k[x_n]/a(x_n)$ is a product of fields, by induction $(I + (a))/(a) \subset (k[x_n]/a(x_n))[x_1, \ldots, x_{n-1}]$ also can be generated up to radical by $n - 2$ elements. Now by the lemma, $I$ can be generated up to radical by $n - 1$ elements.

If $V$ is not equidimensional, the proof is considerably harder, but the main idea remains the same. □

The rest of this paper is devoted to a question of M. P. Murthy [Mu1], who asked whether every locally complete intersection (l.c.i.) subscheme $V \subset \mathbb{A}^n_k$ is a set-theoretic complete intersection. The answer is known to be positive if $\dim V = 1$ [Fe, Sz, Bo, MK].

We generalize this result as follows:

**THEOREM C.** Every l.c.i. subscheme $V \subset \mathbb{A}^n_k$ of constant positive dimension can be defined by $n - 1$ equations.

A proof of this theorem is similar to that of the equidimensional case of Theorem B. □
Moreover, we have obtained the following:

**Proposition.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $I$ a l.c.i. of equidimension $d$ in $B = k[x_1, \ldots, x_n]$ and $n \geq 3d$. Set $A = B/\sqrt{I}$. Then there exists a l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and $J/J \cdot \sqrt{I}$, the reduced conormal module of $J$, is isomorphic to $P \oplus A^{n-2d+1}$, where $P$ is a projective $A$-module of rank $d - 1$ with trivial determinant.

**Sketch of Proof.** By the Ferrand construction [Fe, p. 24; Va, p. 89] we are reduced to the case when the determinant of the conormal module of $I$ is trivial. Let $c_d \in F^d(\text{Spec } A)$ be the top Chern class of the reduced conormal module of $I$, where $F^d(\text{Spec } A)$ is the $d$th component of the Grothendieck $\gamma$-filtration [FL, p. 48]. Since $F^d(\text{Spec } A)$ is divisible, we can write $c_d = (1 - p^d)c$. Let $Q$ be a projective $A$-module of rank $d$ such that $Q - A^d \in F^d(\text{Spec } A)$ and the top Chern class of $Q$ equals $c$. Set $I/I \cdot \sqrt{I} \approx M \oplus Q$, where $M$ is projective of rank $n - 2d \geq d$. We show that there exists a l.c.i. $J_1 \subset I$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of $J_1$ is $M \oplus F(Q)$, where $F(Q)$ is the Frobenius of $Q$. It is straightforward to compute that the top Chern class of $M \oplus F(Q)$ is zero. Taking the iterated Frobenius of $J_1$, if necessary, we obtain, by [Mu2, Theorem 5] an l.c.i. $J$ with required properties. □

**Theorem D.** Let $k$ be any algebraically closed field of characteristic $p > 0$ and $V \subset A^n_k$ a locally complete intersection subscheme of constant dimension $d$, such that $2 \leq d \leq n - 4$. Then $V$ can be defined by $n - 2$ equations. In particular, every locally complete intersection surface in $A^n_k$, where $n \geq 6$, is a set-theoretic complete intersection.

**Proof.** By induction, using the lemma we reduce to the case $d = \dim V = 2$. By the proposition there exists l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of $J$ is free. Now by [MK, Theorem 5] $J$ is $(n - 2)$-generated. □

**Theorem E.** Let $k$ be any algebraically closed field of characteristic 2 and $V \subset A^n_k$ a locally complete intersection subscheme of constant dimension $d$, such that $3 \leq d \leq n - 6$. Then $V$ can be defined by $n - 3$ equations. In particular, every locally complete intersection threefold in $A^n_k$, where $n \geq 9$, is a set-theoretic complete intersection.

**Sketch of Proof.** By an inductive argument with the help of the lemma we are reduced to the case $d = 3$. By the proposition, we get $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of $J$ is $P \oplus A^{n-5}$, where $P$ is projective of rank 2 with trivial determinant. By a special argument in characteristic 2, we then obtain a new l.c.i. ideal $J_1 \subset J$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of $J_1$ is free. Now we are done by [MK, Theorem 5]. □

Complete proofs will appear elsewhere.
REFERENCES


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