THE NUMBER OF EQUATIONS NEEDED TO DEFINE
AN ALGEBRAIC SET

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Let $B$ be a commutative Noetherian ring, $X = \text{Spec} B$ the associated affine scheme, $I \subset B$ an ideal and $V = V(I) \subset X$ the closed subset defined by $I$.

**DEFINITION.** Elements $f_1, \ldots, f_s \in I$ define $V$ set-theoretically (equivalently, $V$ is defined set-theoretically by $s$ equations $f_1 = 0, f_2 = 0, \ldots, f_s = 0$) if $\sqrt{(f_1, \ldots, f_s)} = \sqrt{I}$.

Hilbert’s Nullstellensatz implies that in the case when $B$ is a finitely generated algebra over an algebraically closed field $k$, this definition agrees with the usual one, i.e., all $f_1, \ldots, f_s$ vanish at a $k$-rational point if and only if it belongs to $V$. In the sequel “defined” always means “defined set-theoretically”.

The question we are dealing with here concerns the minimum number of equations needed to define a given $V \subset X$. A classical result that goes back to L. Kronecker [Kr] says that if $B$ is $n$-dimensional, then $n+1$ equations would suffice for every $V \subset X$. Our first theorem describes those $V \subset X$ which can be defined by $n$ equations.

**THEOREM A.** Let $k$ be an algebraically closed field, $X$ a smooth affine $n$-dimensional variety over $k$ with coordinate ring $B$, and $V = V' \cup P_1 \cup P_2 \cup \cdots \cup P_r$ an algebraic subset of $X = \text{Spec} B$, where $V'$ is the union of irreducible components of positive dimensions and $P_1, P_2, \ldots, P_r$ some isolated closed points (which do not belong to $V'$). Then $V$ can be defined by $n$ equations if and only if one of the following conditions holds.

(i) $r = 0$, i.e., $V$ consists only of irreducible components of positive dimension.

(ii) $V'$ is empty, i.e., $V$ consists only of closed points and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1 P_1 + n_2 P_2 + \cdots + n_r P_r = 0$ in $A_0(X)$.

(iii) $V'$ is nonempty, $r \geq 1$ and there exist positive integers $n_1, n_2, \ldots, n_r$ such that $n_1 P_1 + n_2 P_2 + \cdots + n_r P_r$ belongs to the image of the natural map $A_0(V') \to A_0(X)$ induced by the inclusion $V' \to X$.

Here $A_0(\cdot)$ stands for the group of zero-cycles modulo rational equivalence [Fu].

**SKETCH OF PROOF.** Our proof consists of three steps. In Step 1 we construct an ideal $I \subset B$ such that $\sqrt{I}$ is the defining ideal of $V$, and in addition $I$ has some other special properties. In Step 2 we, in a special way, pick some ideals $Q_1, \ldots, Q_n$ such that $\sqrt{Q_i}$ is a maximal ideal containing $I$ for each $i$ and $J/J^2$ is $n$-generated, where $J = I \cap Q_1 \cap \cdots \cap Q_n$. In Step 3 we...
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prove that the 0-dimensional Segre class of \( J \) equals 0, so by [Mu2, Theorem 2] and the Suslin cancellation theorem [Su], \( J \) is \( n \)-generated. □

If \( B \) is not assumed to be regular, the result of Theorem A need not hold. In fact, for every \( n \geq 0 \), we have constructed an example of a finitely generated \( n \)-dimensional algebra \( B \) over a suitable algebraically closed field, such that its singular locus consists of just one closed point and for every \( d \) between 1 and \( n - 1 \) it contains a \( d \)-dimensional subvariety which cannot be defined by \( n \) equations.

Storch [St] and Eisenbud-Evans [EE] proved that every algebraic set in \( \mathbb{A}^n_k \) can be defined by \( n \) equations, while Cowisk-Nori [CN] proved that every curve in \( \mathbb{A}^n_k \), where \( \text{char} k = p > 0 \), can be defined by \( n - 1 \) equations. Our next theorem sharpens these results.

**Theorem B.** If \( \text{char} k = p > 0 \), then every algebraic set \( V \subset \mathbb{A}^n_k \) consisting only of irreducible components of positive dimensions can be defined by \( n - 1 \) equations.

**Sketch of Proof.** The main idea of the proof is contained in the following lemma.

**Lemma.** Let \( f_1, \ldots, f_r, g_1, \ldots, g_r \in I \) and \( a \in B \) such that

(i) \( f_1, \ldots, f_r \) generate \( I_a \) up to radical.

(ii) \( g_1, \ldots, g_r \) generate \( (I + (a))/(a) \subset B/(a) \) up to radical. Then there exist \( r + 1 \) elements \( h_1, \ldots, h_{r+1} \) which generate \( I \) up to radical.

Now assume all irreducible components of \( V \) have dimension \( d \geq 1 \) and use induction on \( d \), the case \( d = 1 \) being settled by [CN]. Let \( I = I(V) \subset B = k[x_1, \ldots, x_n] \) be the defining ideal of \( V \). By a change of variables we can assume that \( k[x_n] \) has zero intersection with every minimal prime overideal of \( I \). By induction the extension of \( I \) in \( k[x_n][x_1, \ldots, x_{n-1}] \) can be generated up to radical by \( n - 2 \) elements. Thus there exists a square-free polynomial \( a(x_n) \) such that \( I_{a(x_n)} \) can be generated up to radical by \( n - 2 \) elements. Since \( k[x_n]/a(x_n) \) is a product of fields, by induction \( (I + (a))/(a) \subset (k[x_n]/a(x_n))[x_1, \ldots, x_{n-1}] \) also can be generated up to radical by \( n - 2 \) elements. Now by the lemma, \( I \) can be generated up to radical by \( n - 1 \) elements.

If \( V \) is not equidimensional, the proof is considerably harder, but the main idea remains the same. □

The rest of this paper is devoted to a question of M. P. Murthy [Mu1], who asked whether every locally complete intersection (l.c.i.) subscheme \( V \subset \mathbb{A}^n_k \) is a set-theoretic complete intersection. The answer is known to be positive if \( \dim V = 1 \) [Fe, Sz, Bo, MK].

We generalize this result as follows:

**Theorem C.** Every l.c.i. subscheme \( V \subset \mathbb{A}^n_k \) of constant positive dimension can be defined by \( n - 1 \) equations.

A proof of this theorem is similar to that of the equidimensional case of Theorem B. □
Moreover, we have obtained the following:

**Proposition.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), \( I \) a l.c.i. of equidimension \( d \) in \( B = k[x_1, \ldots, x_n] \) and \( n \geq 3d \). Set \( A = B/\sqrt{I} \). Then there exists a l.c.i. \( J \subset I \) such that \( \sqrt{J} = \sqrt{I} \) and \( J/J \cdot \sqrt{I} \), the reduced conormal module of \( J \), is isomorphic to \( P \oplus A^{n-2d+1} \), where \( P \) is a projective \( A \)-module of rank \( d - 1 \) with trivial determinant.

**Sketch of Proof.** By the Ferrand construction [Fe, p. 24; Va, p. 89] we are reduced to the case when the determinant of the conormal module of \( I \) is trivial. Let \( c_d \in F^d(\text{Spec } A) \) be the top Chern class of the reduced conormal module of \( I \), where \( F^d(\text{Spec } A) \) is the \( d \)th component of the Grothendieck \( \gamma \)-filtration [FL, p. 48]. Since \( F^d(\text{Spec } A) \) is divisible, we can write \( c_d = (1-p^d)c \).

Let \( Q \) be a projective \( A \)-module of rank \( d \) such that \( Q - A^d \in F^d(\text{Spec } A) \) and the top Chern class of \( Q \) equals \( c \). Set \( I/I \cdot \sqrt{I} \approx M \oplus Q \), where \( M \) is projective of rank \( n - 2d \geq d \). We show that there exists a l.c.i. \( J_1 \subset I \) such that \( \sqrt{J_1} = \sqrt{I} \) and the reduced conormal module of \( J_1 \) is \( M \oplus F(Q) \), where \( F(Q) \) is the Frobenius of \( Q \). It is straightforward to compute that the top Chern class of \( M \oplus F(Q) \) is zero. Taking the iterated Frobenius of \( J_1 \), if necessary, we obtain, by [Mu2, Theorem 5] an l.c.i. \( J \) with required properties. \( \Box \)

**Theorem D.** Let \( k \) be any algebraically closed field of characteristic \( p > 0 \) and \( V \subset \mathbb{A}^n_k \) a locally complete intersection subscheme of constant dimension \( d \), such that \( 2 \leq d \leq n - 4 \). Then \( V \) can be defined by \( n - 2 \) equations. In particular, every locally complete intersection surface in \( \mathbb{A}^n_k \), where \( n \geq 6 \), is a set-theoretic complete intersection.

**Proof.** By induction, using the lemma we reduce to the case \( d = \dim V = 2 \). By the proposition there exists l.c.i. \( J \subset I \) such that \( \sqrt{J} = \sqrt{I} \) and the reduced conormal module of \( J \) is free. Now by [MK, Theorem 5] \( J \) is \((n-2)\)-generated. \( \Box \)

**Theorem E.** Let \( k \) be any algebraically closed field of characteristic \( 2 \) and \( V \subset \mathbb{A}^n_k \) a locally complete intersection subscheme of constant dimension \( d \), such that \( 3 \leq d \leq n - 6 \). Then \( V \) can be defined by \( n - 3 \) equations. In particular, every locally complete intersection threefold in \( \mathbb{A}^n_k \), where \( n \geq 9 \), is a set-theoretic complete intersection.

**Sketch of Proof.** By an inductive argument with the help of the lemma we are reduced to the case \( d = 3 \). By the proposition, we get \( J \subset I \) such that \( \sqrt{J} = \sqrt{I} \) and the reduced conormal module of \( J \) is \( P \oplus A^{n-5} \), where \( P \) is projective of rank \( 2 \) with trivial determinant. By a special argument in characteristic \( 2 \), we then obtain a new l.c.i. ideal \( J_1 \subset J \) such that \( \sqrt{J_1} = \sqrt{I} \) and the reduced conormal module of \( J_1 \) is free. Now we are done by [MK, Theorem 5]. \( \Box \)

Complete proofs will appear elsewhere.
REFERENCES


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