ON THE LOCAL SEVERI PROBLEM

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Introduction. We study plane curves with singularities. Let \( P^N \) be the projective space parametrizing plane curves of degree \( n \) \((N = n(n+3)/2)\). Let \( V(n, g) \subset P^N \) be the locus of reduced irreducible plane curves of degree \( n \) and (geometric) genus \( g \), and \( l \subset P^2 \) a fixed line. Following Zariski [7], we consider the subvariety \( Z(n, g) \subset \overline{V}(n, g) \) of curves which contain \( l \) as a component. The purpose of this note is to study \( Z(n, g) \) and prove the following

\[ \text{Theorem.} \] Let \( E(n, g) \) be a branch of \( \overline{V}(n, g) \) through a point of \( Z(n, g) \) corresponding to a reduced curve. Then the general members of \( E(n, g) \cap Z(n, g) \) have only nodes as singularities.

It is well known (cf. Severi [5, §11]) that this Theorem implies the following fundamental result of Harris.

\[ \text{Corollary (Harris [3]). } V(n, g) \text{ is irreducible.} \]

In the case when \( L \in E(n, g) \cap Z(n, g) \) is a union of \( n \) distinct lines passing through a point, our theorem is a realization of Severi’s attempt to prove that \( L \) can be regenerated to a reducible nodal curve of \( E(n, g) \) [5, §11, p. 344]. The idea of using decreasing induction on \( g \) and equations of curves in the proof was suggested in Zariski [7]. On the other hand, Harris [3] and Ran [4] use the degeneration method in their treatment of plane curves.

Proof of Theorem. We set \( d = (n - 1)(n - 2)/2 - g \) and \( \nu(n, d) = \dim V(n, g) = 3n + g - 1 \) ([5, §11], [6]). Let \( \Sigma_{n,d} \subset P^N \times \text{Sym}^d(P^2) \) be the closure of the locus of irreducible curves of degree \( n \) with \( d \) nodes and no other singularities, and \( \pi_N \) the projection to \( P^N \). Given a pair consisting of a reduced curve \( E \in \overline{V}(n, g) \) and a branch of \( \overline{V}(n, g) \) through the curve, one can define, via \( \pi_N \), an element of \( \text{Sym}^d(P^2) \), called the cycle of assigned singularities of the pair. Our basic tool is the dimension-theoretic characterization of maximal families of nodal curves by Arbarello and Cornalba [1] and Zariski [6] and its generalization by Harris [3, Proposition 2.1].

Let \( C \) be a general member of \( E(n, g) \cap Z(n, g) \). We will prove that \( C \) is nodal and all its unassigned nodes lie on \( l \) for every choice of a branch of \( E(n, g) \) through \( C \).

Lemma. For \( d \leq 3 \), \( \Sigma_{n,d} \) is irreducible and unibranch.

Proof of the Lemma. Let \( \Sigma' \), \( \Sigma'' \subset \Sigma_{n,d} \) be components such that a general member of \( \Sigma' \) has \( d \) nodes in general position. A dimension count

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shows that a general member of $\Sigma' \cap \Sigma''$ is a reduced curve with $d$ assigned singular points. Therefore $\Sigma_{n,d} = \Sigma'$. The second assertion follows from the unibranchness of $\text{Sym}^d(\mathbb{P}^2)$.

**Step 1.** Let $d \leq 3$. We regenerate $C$ to a nodal curve $F = l + F'$ having the same number of irreducible components and the same genus as $C$. We take a branch of $\mathcal{C}(n, g)$ through $C$ and consider the corresponding cycle of assigned singularities $\sum d_t P_t$. For every $t$, we choose $d_t$ assigned nodes of $F$ in the vicinity of $P_t$. The curve $F$ with the $d = \sum d_t$ assigned nodes determines a branch of $\overline{\mathcal{V}(n, g)}$ through $C$. By the lemma, it coincides with the branch of $\mathcal{C}(n, g)$, chosen above.

**Step 2.** We assume $d \geq 4$ and the theorem is true for smaller $d$'s. If $C = l + C'$ has no assigned singularities on $l$ (for a branch of $\mathcal{C}(n, g)$), then $C'$ is moving in the family of dimension $\leq \nu(n - 1, d) = \nu(n, d) - n - 1$. Since $Z(n, g)$ is defined by $n + 1$ equations in $\mathcal{C}(n, g)$ [7, p. 470], the inequality is, in fact, equality and $C$ is nodal.

**Step 3.** We now assume that $C$ has assigned singularities on $l$. Let $f(X, Y, Z) = \Sigma a_{ijk} X^j Y^k Z^{n-j-k}$ be an equation of a curve of $\mathcal{C}(n, g)$. We have chosen our coordinate system in $\mathbb{P}^2$ so that $l = \{X = 0\}$ and all the singularities of the curves of $\mathcal{C}(n, g)$ lie in $\mathbb{P}^2 \setminus \{Z = 0\}$. Moreover, the following constructions take place in a neighborhood of a general member $D$ of $\mathcal{C}(n, g) \cap Z(n, g) \cap \{a_{00} = a_{10} = a_{01} = 0\}$. We assume $D$ is a specialization of $C$, and it has an assigned singularity at $(0:0:1)$. By abuse of notation we denote by $\mathcal{C}(n, g)$ a fixed new branch of $\overline{\mathcal{V}(n, g)}$ through $D$, contained in the original branch.

Let $\mathcal{A}_g^0 \subset \mathcal{C}(n, g)$ be the subfamily of curves having a node at $(0:0:1)$. It has codimension 2 in $\mathcal{C}(n, g)$ and its general members are irreducible nodal curves with $d$ nodes. For $i \geq 1$, the components of $\mathcal{A}_g^i = \mathcal{A}_g^{i-1} \cap \{a_{0n-i+1} = 0\}$ have codimension $\leq i + 2$ in $\mathcal{C}(n, g)$ and consist of curves having intersection multiplicity at least $i$ with $l$ at $(0:1:0)$. If a general member of $\mathcal{A}_g^i$ does not contain $l$ as a component, then a dimension count shows that it has intersection multiplicity $i$ with $l$ at $(0:1:0)$ and only $d$ singular points which are nodes: we blow up $(0:0:1)$ and $i$ times $(0:1:0)$ in the direction of $l$, and apply [3, Proposition 2.1].

**Step 4.** Let $E$ be a general member of $\mathcal{C}(n, g)$ and $Q \in E$ a node distinct from $(0:0:1)$. Moreover, if $D$ has an assigned (with respect to $\mathcal{C}(n, g)$) singular point outside $l$, then we assume $Q$ tends to this point; the second choice is a node $Q \in E$ which does not tend to $(0:0:1)$. Let $\mathcal{C}(n, g + 1) \subset \overline{\mathcal{V}(n, g + 1)}$ ($\mathcal{C}(n, g) \subset \mathcal{C}(n, g + 1)$) be the branch through $D$ obtained by considering $Q$ as virtually nonexistent. We can define, as above, the subfamilies $\mathcal{A}_g^{i+1}$ of $\mathcal{C}(n, g + 1)$. Let $m + 1$ be the first integer such that a general member of $\mathcal{A}_g^{m+1}$ contains $l$ as a component.

**Case 1.** The general members of $\mathcal{A}_g^{m+1}$ do not contain $l$ as a component. We get $\dim \mathcal{A}_g^{m+2} = \dim \mathcal{A}_g^{m+1}$. By the induction hypothesis we get that $D$ is nodal.

**Case 2.** A general member $F$ of $\mathcal{A}_g^{m+1}$ contains $l$ as a component. Then $F = l + F'$ is nodal. A dimension count shows that $D = l + D'$ can have at
most one non-nodal singularity. Moreover, if \( g(F') \neq g(D') \) then \( D \) is nodal. If \( D \) is not nodal and \( g(F') = g(D') \), then the singular points of \( F \) and \( D \) are on \( l \). Hence \( F' \) is smooth and \( D \) has one tacnode. We can assume the node \( Q \) tends to a node \( Q^* \in D \). Let \( \mathcal{H} \subset \Sigma_{n,1} \) be the branch through \((D, Q^*)\). By [2, Exp. XIII, §2],

\[
\mathcal{H} = \mathcal{E}(n, g + 1) \cap \pi_N(\mathcal{H})
\]

is 1-connected. If \( \mathcal{H} = \mathcal{E}(n, g) \), we are done. If \( \mathcal{H} \neq \mathcal{E}(n, g) \), we choose a component \( \mathcal{C} \) of \( \mathcal{H} \) such that a component \( \mathcal{U} \) of \( \mathcal{C} \cap \mathcal{E}(n, g) \) has dimension \( \nu(n, d + 1) \). Let \( G \) be a general member of \( \mathcal{U} \). By the deformation theory, we get \( g(G) \leq g - 1 \). Therefore \( G \) is nodal with \( d + 1 \) nodes, two of which tend to the tacnode. By intersecting \( \mathcal{U} \) with the branches of \( \pi_N(\Sigma_{n,1}) \) corresponding to the unassigned nodes of \( D \), we derive that \( D \) is not a general curve.

**REMARKS.** A dimension count shows that the number of unassigned singularities of \( C \) is equal to \( m \).

As in the lemma, for \( d \leq n(n + 3)/6 \) and \((n, d) \neq (6, 9)\), the unibranchness of \( \Sigma_{n,d} \) follows from the irreducibility. One can give another proof that \( \Sigma_{n,d} \) is irreducible in that range using the following general result (see a conjecture in [7, p. 479]): Let \( E \) be a general member of an irreducible subfamily \( S \) of \( \mathcal{V}(n, g) \) \((0 \leq g \leq (n - 1)(n - 2)/2) \). If \( \dim S \geq \nu(n, d) - 1 \), then \( E \) is reduced. If \( S \) consists of nonreduced curves and has the maximal dimension, then a general member of \( S \) is a union of an irreducible nodal curve and a general double line.

**REFERENCES**


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