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Ill-posed problems of mathematical physics and analysis, by M. M. Lavrent′ev, V. G. Romanov, and S. P. Shishatskiĭ; translated by J. R. Schulenberger. Translations of Mathematical Monographs, vol. 64, American Mathematical Society, Providence, R. I., 1986, vi + 290 pp., $98.00. ISBN 0-2818-4517-9

As is well known, a problem is said to be well-posed in the sense of Hadamard when a unique solution exists and depends continuously upon the data. The definition is made precise by stipulating not only the function spaces in which the solution and data are to lie but also the measures and notion of continuity. A problem that is not well-posed is said to be ill-posed.

Although nineteenth-century mathematicians contributed to the early study of ill-posed problems, it is generally agreed that the subject came to prominence only after Hadamard had formulated his well-known definition. Unfortunately, he developed an adverse view of the subject which, on becoming widely accepted, had the effect of inhibiting further study. His objections were grounded in his celebrated counterexample of the Cauchy problem for Laplace's equation. In order for there to be global existence of the solution Hadamard demonstrated that the Cauchy data must satisfy a certain compatibility relation but even in the unlikely event of the relation being satisfied he further showed that the solution in general does not depend continuously on the data. Such behaviour convinced Hadamard that ill-posed problems lacked physical relevance and hence should be ignored. This became the prevailing attitude, and consequently, in partial differential equations at least, activity became confined to the standard initial boundary value problems. It was only the growing insistence for a precise theoretical understanding from the applied sciences, principally geophysics and computing, that rekindled mathematical interest.

It is worth considering briefly why ill-posed problems are of practical importance and hence merit detailed study. Take, for example, the simple Dirichlet problem for a linear elliptic homogeneous differential equation. Conditions are known guaranteeing that the solution exists, is unique and depends continuously upon the Dirichlet data, i.e., the problem is well-posed. These conditions include the requirement that the solution be specified in a suitable sense at all points of the boundary of the region of definition. Yet, rarely, if ever, can this specification be completely achieved in practice. Measuring devices record only approximate values and in any case are able to measure data only at a discrete number of points and not over the entire boundary as demanded by the mathematical theory. The solution is therefore not uniquely determined by the measured values of the data and consequently cannot depend continuously on them. Thus, when subjected to the limitations of the measuring device, even standard problems in differential equations are liable to be ill-posed.

There are, of course, many other examples of ill-posed problems. In differential equations the most frequently cited include not only the Cauchy problem for Laplace's equation but also the backward heat equation, the Dirichlet problem for the wave equation, and the wave and parabolic equations subject to data on time-like regions. Lest these be seen as somewhat contrived, it must be emphasised that they all serve as models for practical problems. For instance, the Cauchy problem for Laplace's equation corresponds to the situation, encountered in geophysics, surveying and mineral prospecting, where only part of the boundary is accessible for the measurement of data, but over which an abundance of data can be collected. Further examples arise from inaccuracies in the measurement of the geometry of the region of definition and also the value of the operator. Again, the coefficients themselves in the differential equations and boundary operators are part of the data and as such are also subject to measurement errors. Lack of precision in determining the coefficients casts doubt on the validity of supposing that an equation is of definite type, so that, strictly speaking, a comprehensive theory of ill-posed problems should also include differential equations of indefinite type.

Ill-posed problems also occur in many other branches of mathematics, elementary examples being the Fredholm integral equation of the first kind, analytic continuation of a function, determination of the derivative of a function that is only approximately specified, and a singular linear system of algebraic equations. A further important class concerns inverse problems where it is typically required to determine the coefficients of an equation from a knowledge of certain functionals of the solution. A well-known example is the one-dimensional inverse Sturm-Liouville problem, in which the value of the ordinary differential operator is to be determined from the spectral function of the solution. Other examples arise in inverse scattering theory, while of increasing significance are problems with free boundaries.

Reference has already been made to one practical situation producing an ill-posed problem. Others are found in a wide range of disciplines including medicine, continuum mechanics, control theory, nondestructive testing of materials, and meteorology. Indeed, given our previous remarks it can be argued with only slight exaggeration that most problems arising in practice are inevitably ill-posed. (Additional examples of ill-posed problems and the contexts in which they arise are described in the expository survey by Payne **[3]**; see also Tikhonov and Arsenin **[5]** and the opening chapter of the book under review.)

A common feature of many ill-posed problems is that the solution possesses the kind of instability exhibited in Hadamard's original counterexample. The aim is therefore to establish conditions under which the problem may be stabilised and continuous dependence on the data, and hence uniqueness, thereby recovered. Generally speaking, this is achieved by following two ideas first introduced by Pucci and F. John. The first is to impose a suitable constraint on the class of admissible solutions, such as requiring them to be bounded in norm. The second is to relax the notion of continuity employed to that of, say, either Hölder or logarithmic continuity.

Since the 1950s a vast literature has been devoted to these questions. At first, major concern was with uniqueness, but then attention became increasingly concentrated on the construction of methods yielding continuous dependence. There is now available a large variety of such methods, including those based upon eigenfunction expansions, function-theoretic methods, logarithmic convexity and similar arguments, weighted energy, the Lagrange identity, comparison arguments, and convergent integrals. Since the error in the data may not necessarily be deterministic, other methods have also been developed which are probabilistic in nature. A good account of most of the techniques employed may be found in Payne [**3**]; see also Straughan [**4**]. However, it is not yet entirely clear what particular approach is best suited to a given class of problems. For instance, the technique based upon the Lagrange identity appears limited to first- and second-order linear autonomous self-adjoint differential equations, whereas methods involving logarithmic convexity or related arguments may be applied to more general classes of equations. Both approaches yield continuous dependence in the sense of Hölder, but in those cases where they both apply, the one based upon the Lagrange identity often requires less stringent hypotheses on the data and regularity of the solution. Nevertheless, as the various techniques become further refined the distinction between them appears to be becoming increasingly blurred, suggesting perhaps that there should be renewed effort to fully understand the basic structure not only of the techniques themselves, but also of the fundamental theory of ill-posed problems. Naturally, elements of an abstract theory have been known for some time. For instance, Tikhonov has shown that a large class of ill-posed problems satisfy a modified definition of well-posedness in which existence and uniqueness are assumed, but in which continuous dependence is required to hold only on some subspace, usually taken to be compact. The latter condition corresponds to the constraint which, as described earlier, is imposed when stabilising the problem, while the abstract notion of continuity incorporates the relaxation of the continuity concept.

Much of the work undertaken in ill-posed problems has been stimulated by demands arising from the numerical evaluation of solutions where it is vital to have conditions guaranteeing continuous dependence of the solution on the data or input. The essential finite-dimensional nature of all numerical schemes combined with unavoidable rounding errors associated with hardware means that many of the difficulties common to ill-posed problems are quickly encountered in computation. Their resolution has greatly contributed to the general advance of the subject as a whole.

So far, we have not mentioned the question of existence, which has been less extensively studied than either uniqueness or continuous dependence. It is characteristic of ill-posed problems, and especially of those connected with the Cauchy problem for differential equations, that a solution will not exist globally even for arbitrary smooth data. Thus, work in this area has centred largely on the notion of "best-approximate" solution in the sense that there is some function which in a suitable space and with respect to an appropriate measure best approximates the associated data and equation. In the method of quasireversibility, due originally to Lattes and Lions [**2**], the idea is to penalise the actual equation by the addition of a term rendering the problem well-posed and then to examine the limiting behaviour as the additional term is allowed to vanish. Of course, this idea is found elsewhere, including the study of nonlinear wave equations, where it is known as the artificial viscosity method. Another technique used to study existence is Tikhonov regularisation (see, e.g., [**5**] and the present book). Here, an operator is constructed which when acting on the approximate data yields a solution stable to small perturbations in the data and which also tends to the actual solution as the data tends to its exact value. Both the methods of quasireversibility and regularisation have been extended and refined but apart from them there have been few other major developments. On the other hand, by means of concavity and other arguments, several conditions are now known under which solutions to nonlinear differential equations fail to exist globally.

Despite the immense activity in the study of ill-posedness, surprisingly few monographs or texts have appeared, probably because of the diverse directions the subject has taken and the lack of any overall basic theory. Techniques have tended to be developed for different special classes of problems and in any case the notion of ill-posedness pervades such disparate fields of application that any neat codification probably cannot be expected. The mammoth task of kneading the bewildering array of material into a definitive account of theory, technique and example is, moreover, continually frustrated by the appearance of numerous fresh results.

The present authors, all distinguished mathematicians in the field, are therefore wise to steer a middle course. Their book—first published in Russian in 1980—eschews attempts to be encyclopaedic and instead divides its attention between fundamentals and a discussion of several particular classes of ill-posed problems. While little of the abstract theory is new, covering ground that may be found, for instance, in the books by Tikhonov and Arsenin [**5**] and by Lavrentiev [**2**], the opportunity is taken to update and expand the material, which results in a comprehensive introduction to some basic analytic concepts, including the method of regularisation, and a good discussion of the properties of the ill-posed (nonlinear) operator equation $Ax = f$ where A has unbounded inverse. These introductory principles are then immediately illustrated by problems concerned with analytic continuation of a function of several complex variables, a vital ingredient here being the familiar Hadamard three circles theorem.

However, the major part of the book consists of an extended treatment, based largely on Russian contributions, of particular classes of problems. There are two broad groups, both concerned with partial differential equations. The first deals with parabolic and hyperbolic equations subject to Cauchy data on time-like regions, and also discusses analogous problems for elliptic and ultrahyperbolic equations. This is an important area of comparatively recent origin, and so a detailed account is of especial interest. The general technique employs a weight function to establish estimates which at once yield continuous dependence for suitably constrained solutions. Uniqueness immediately follows. The manipulations, elementary but prolix, occupy several pages and perhaps much of the detail could have been omitted. Nevertheless, it is salutary to sometimes be explicitly reminded of the intricate arguments required to complete these calculations, besides which the account also serves as a good guide to those wishing to treat other problems by a similar method.

The discussion of the second group of problems occupies the final three chapters and is unified by the study of methods for the stabilisation of integral equations of the first kind. The first two of these chapters, both important in their own right, deal with ill-posed problems respectively arising in Volterra integral equations and integral geometry. Integral geometry is concerned with the determination of a function from its given (weighted) mean value over specified manifolds and includes, for example, the well-known Radon transform and the theory of spherical means. Several stability results are obtained for manifolds of fairly general type and further results are presented in the restrictive cases where, for example, analyticity of both manifold and solution is introduced. A separate discussion is devoted to the group of problems in which the manifolds become a family of smooth hypersurfaces contained in a sufficiently small region where it is required to determine the solution. The approach adopted in most of these problems is to rewrite the given mean value equation as a Volterra equation and then exploit properties derived in the previous chapter. The rewriting is not always straightforward and often involves considerable ingenuity.

The final chapter is devoted to multidimensional inverse problems involving second-order parabolic and hyperbolic equations and also first-order hyperbolic systems. The treatment, carried out at both the abstract and concrete level, leads to the determination of both the coefficients of the equation and the value of the operator from a knowledge of the solution on certain manifolds such as the plane $x_3 = 0$. The general method relies upon conversion to an integral equation of the first kind whose kernel is then analysed, often again by extremely delicate arguments, in order to reduce the problem to one of integral geometry.

Synopses of related work, together with bibliographical references mainly to Russian publications, help to place the material of each chapter in proper perspective. The explanations throughout are clear and the pace of the book maintained by the introduction of new examples at each stage. The result is a highly readable, well-organised, coherent account of important selected topics in a major field of contemporary research. The translation is excellent but the absence of an index regrettable. The book, which is a welcome addition to the literature, should prove of value not only to the specialist but also to those requiring a reliable general introduction to the entire subject.

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The equations of Navier-Stokes and abstract parabolic equations, by Wolf von Wahl. Friedr. Vieweg & Sohn, Braunschweig and Wiesbaden, xxiv + 264 pp. (paperback). ISBN 3-528-08915-6

The Navier-Stokes equations are the fundamental equations governing the motion of viscous fluid. Among the versions of these equations, we consider here the nonstationary Navier-Stokes equations for viscous *incompressible* fluid. The system of equations is a nonlinear parabolic equation. A fundamental analytic question would be whether or not a unique regular solution exists for all time for given initial data. This problem was first attacked by Leray more than 50 years ago. It turns out that the situation is quite different depending on the space dimension of the domain Ω occupied with fluid where the unknown functions are defined. When the space dimension is two, a unique regular solution exists for all time provided that the initial data satisfy a compatibility condition and have finite energy. However, when the space dimension is three, the problem is still fundamentally open. We do not know in general whether a regular solution exists for all time even if the initial data are smooth and the domain Ω has no boundary.

In his famous pioneering paper published in Acta Mathematica in 1934, Leray studied the nonstationary Navier-Stokes equations on the three dimensional Euclidean space $\mathbf{R}^3$. He showed:

(I) existence of a global-in-time weak solution satisfying the energy inequality,

and studied its regularity. Once such a weak solution was shown to be regular, the problem could be solved. This method is by now well known especially for solving variational problems. Let us briefly review his idea for studying regularity of his weak solution. He showed:

(II) existence of a unique local-in-time regular solution with nonregular initial data.

At almost all times t_0, the weak solution is in $L^p(\Omega)$ $(2 \leq p \leq 6)$ but this does not directly imply that the weak solution is smooth at t_0. (II) gives a local regular solution starting from time t_0 which is initially, the same as the value