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R. J. KNOPS

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The equations of Navier-Stokes and abstract parabolic equations, by Wolf von Wahl. Friedr. Vieweg & Sohn, Braunschweig and Wiesbaden, xxiv + 264 pp. (paperback). ISBN 3-528-08915-6

The Navier-Stokes equations are the fundamental equations governing the motion of viscous fluid. Among the versions of these equations, we consider here the nonstationary Navier-Stokes equations for viscous *incompressible* fluid. The system of equations is a nonlinear parabolic equation. A fundamental analytic question would be whether or not a unique regular solution exists for all time for given initial data. This problem was first attacked by Leray more than 50 years ago. It turns out that the situation is quite different depending on the space dimension of the domain Ω occupied with fluid where the unknown functions are defined. When the space dimension is two, a unique regular solution exists for all time provided that the initial data satisfy a compatibility condition and have finite energy. However, when the space dimension is three, the problem is still fundamentally open. We do not know in general whether a regular solution exists for all time even if the initial data are smooth and the domain Ω has no boundary.

In his famous pioneering paper published in Acta Mathematica in 1934, Leray studied the nonstationary Navier-Stokes equations on the three dimensional Euclidean space $\mathbf{R}^3$. He showed:

(I) existence of a global-in-time weak solution satisfying the energy inequality,

and studied its regularity. Once such a weak solution was shown to be regular, the problem could be solved. This method is by now well known especially for solving variational problems. Let us briefly review his idea for studying regularity of his weak solution. He showed:

(II) existence of a unique local-in-time regular solution with nonregular initial data.

At almost all times t_0, the weak solution is in $L^p(\Omega)$ $(2 \leq p \leq 6)$ but this does not directly imply that the weak solution is smooth at t_0. (II) gives a local regular solution starting from time t_0 which is initially, the same as the value of the weak solution at t_0. This shows that the weak solution is regular for a short time after t_0 provided that uniqueness holds, namely,

(III) the weak solution agrees with a regular solution provided that the latter exists for the same initial data.

The life span of the regular solution becomes large if the size of the solution at t_0 is small. Applying (II) and (III), one observes that the weak solution must be already large near the singular time where the solution becomes singular, since otherwise the regular solution could be extended beyond the singular time and the weak solution would be smooth near the singular time. On the other hand, the energy inequality for the weak solution implies that some integrals of the modulus of the solution are finite. If there were a large set of singular times, even such integrals would be infinite since the weak solution is already large near the singular time. We thus observe that the set of singular times should not be very large. Such a result is called

(IV) partial regularity of the weak solution.

In fact, Leray proved that his weak solution is regular except at a set of singular times whose 1/2-Hausdorff measure is zero. When the domain Ω has boundary, extra difficulty arises in each step. Result (I) was established by Hopf in 1951 by Galerkin's method.

The book under review focuses on (II), (III), and some part of (IV) for the initial-boundary value problem of the Navier-Stokes equations on smoothly bounded domain Ω in $\mathbf{R}^n$ $(n \geq 2)$ with homogeneous Dirichlet boundary condition. There are essentially two methods for studying (II)—existence of a unique local regular solution for nonregular initial data. The first one is called the energy method, which is explained, for example, in a book of Temam (**1977**). The second one is called the semigroup approach and is based on estimates for solutions of the Stokes equations—the linearized Navier-Stokes equations around the zero solution.

The book under review depends on the semigroup approach and does not use the energy method, except for the energy inequality. We should note that Leray's method is essentially a semigroup approach. In his case, since $\Omega = \mathbf{R}^3$ has no boundary, the Stokes equations are equivalent to the heat equation. Using estimates for the Gaussian kernel, Leray constructed a local solution by a successive approximation. The nonlinear term is considered as a perturbation term. Compared with the energy method, one can prove the existence of solutions under *weaker* regularity assumptions on initial data. This is not only interesting by itself but also important in applying (III) and (IV). If Ω has a boundary, the fundamental solution of the Stokes equations cannot be written explicitly in terms of elementary functions such as the Gaussian kernel, and the Stokes equations are no longer equivalent to the heat equation. The theory of analytic semigroups is useful to overcome this difficulty.

Many initial boundary value problems of partial differential equations such as the heat and Stokes equations can be reformulated as an ordinary equation

(1) $$du/dt + Au = 0, \qquad u(0) = u_0,$$

in Hilbert or Banach spaces, where A is an unbounded operator. For example, if the original equation is the heat equation, $-A$ is the Laplace operator with homogeneous boundary condition. When (1) represents the Stokes equation, A is called the Stokes operator. The solution is formally written as $u = e^{-tA}u_0$, where e^{-tA} is called a semigroup of operators. Semigroup theory studies e^{-tA} using the resolvent of A. This idea goes back to the Laplace transformation. The theory is successful if A is linear. The theory of analytic semigroups is especially suited for parabolic equations, namely, the case when A is an elliptic operator. In the early 1960s Fujita and Kato applied analytic semigroup theory to construct a local regular solution for the initial boundary value problem of the Navier-Stokes system. They transformed the system to an abstract evolution equation

$$du/dt + Au = Fu, \qquad u(0) = u_0, \tag{2}$$

where Fu corresponds to the nonlinear term in the system and A is the Stokes operator. They solved (2), regarding Fu as a perturbation term. The space they were working on is a Hilbert space contained in L^2. To estimate the nonlinear term Fu, they used fractional powers of A. An extension to Banach spaces was announced by Sobolevskiĭ (**1959**).

The book studies the Navier-Stokes equations as an abstract parabolic evolution equation of the form (2) (with external force) in Banach spaces contained in L^p for $1 < p < \infty$. For the Stokes operator, the analyticity of e^{-tA} in such a space was shown by Solonnikov (**1973**) and the reviewer (**1981**) by different methods. The necessary theory of fractional powers of A was established by the reviewer (**1985**). Based on these, the author gives various results of local existence for the Navier-Stokes equation (2) as well as abstract parabolic equations, where initial data are taken as nonregular as possible. This effort is useful to improve (III) and (IV).

(III) was systematically studied by Serrin (**1963**). He proved that if one weak solution belongs to the suitable Lebesgue space, then two weak solutions agree provided that initial data is the same. The book improves Serrin's result for uniqueness by applying results related to (II) and an approximation argument. The study of (IV) is developing in two ways. One way is to study partial regularity of weak solutions which are known to exist. The other way, which was started by Serrin (**1962**), is to give simple criteria for the global regularity of the weak solution. The book also improves such criteria, using results related to (II) and Solonnikov's a priori L^p estimate for the Stokes equations. One of the typical results is that if the velocity field $u(\cdot, t)$ is considered as a continuous curve in $L^n(\Omega)$ then u is regular provided that u solves (2). The weak solutions belonging to the uniqueness class in the book may not be regular. The book studies partial regularity of such solutions.

The organization of the book is as follows. Starting with analytic semigroup theory in Chapter I, the author gives various results of local existence of solutions for abstract nonlinear parabolic equations in Chapter II. The first two sections of Chapter III are devoted to reviewing results on the Stokes operator. The author presents results on local existence as well as global existence for small data for the Navier-Stokes equations. This is the main goal of Chapter III and it elaborates results (II). Chapter IV is devoted to questions related to (III) and (IV). Chapter V studies local and global existence of solutions for abstract parabolic equations by a different method. The author applies these results to semilinear parabolic equations and the Navier-Stokes equations. This provides another proof for regularity criteria in Chapter IV. The major part of Chapters IV and V is based on recent results of the author and Sohr. Each chapter has a section for comments on related results which is very helpful for the reader. Most of prerequisites are given in the book, with or without proof. Finally, the reviewer acknowledges that the book is well proofread, although there are a lot of complicated formulas with sub- and superscripts.

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YOSHIKAZU GIGA

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Differential geometry and topology, by A. T. Fomenko; translated by D. A. Leites. Consultants Bureau, New York and London, 1987, xiii + 323 pp., $ 75.00. ISBN 0-306-10995-6

Most mathematicians soon acquire the habit of scribbling diagrams and drawing pictures when reading mathematics to help them follow the argument. This book, however, contains so many useful and attractive diagrams (234 figures in 323 pages) that scribbling is much less necessary than usual. It is a book for readers who like thinking in pictures better than following a proof through line by line. Explanations are stressed more than formal proofs, theorems which are not needed but are considered interesting are stated without