and implications of the obstruction are rapidly giving rise to entirely new lines of research. This was already outlined by Almeida and Molino [**3**] and, quite recently, confirmed by Weinstein [**10**]. The publication of the book is therefore very opportune and timely.

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Inverse spectral theory, by Jürgen Pöschel and Eugene Trubowitz. Academic Press, Orlando, 1987, x + 192 pp., $29.95. ISBN 0-12-563040-9

Inverse spectral theory and its close relative inverse scattering theory form an active branch of mathematics. The basic question addressed is: given certain data of spectral or scattering type, can one deduce what the underlying operator which governs the process is? Most commonly this becomes the question of determining an unknown coefficient function in operators of a given form. The unknown might be a potential or a mass distribution or even the shape of a scattering obstacle. Such problems are notoriously hard. They are sometimes ill-posed and the solutions may not be unique. Typically the problem has two parts: (I) to characterize the set of data which might arise from a given class of operators and (II) to find which coefficients give rise to a particular admissible data point.

The volume under review, *Inverse spectral theory* by J. Pöschel and E. Trubowitz, gives a complete solution to the inverse problem for the regular Sturm-Liouville operator on an interval with Dirichlet boundary conditions. This means that for the class of ordinary differential operators $-d^2/dx^2+q(x)$ on $[0,1]$ with zero boundary conditions at each end-point, one may characterize the possible eigenvalues $\mu_1 < \mu_2 < \cdots$ and also exhibit unknowns $q(x)$ which give rise to an admissible sequence $\{\mu_j\}$. These results are quite explicit and beautiful. Many of the basic ideas are readily accessible to nonexperts, as will be apparent below. Despite its marketable title, *Inverse spectral theory* restricts itself to the inverse spectral problem for the Sturm-Liouville operator with zero boundary conditions. Rather than surveying the broader subject area, which would require a book or more, this review will follow suit and restrict itself to the material covered by Pöschel and Trubowitz.

The complete solution of the inverse problem described above yields a system of global coordinates on $L^2[0,1]$. These are the eigenvalues $\{\mu_n\}$ and certain other numbers $\{\kappa_n\}$ described below. The level sets of the spectrum $\mu_1 < \mu_2 < \cdots$ form a submanifold of L^2 with a unique point $p(x)$ which is even $(p(1-x) = p(x))$, corresponding to $\kappa_n = 0$, $n = 1, 2, \ldots$.

Three important ingredients in the solution of the inverse problem are described below. The first is the characterization of the spectrum. The second is the calculation of the way in which the eigenvalues μ_n vary with the potential $q(x)$. The third is the existence of Darboux transformations which change $q(x)$ without changing μ_n and other mappings which move a single eigenvalue while fixing all the κ_j and the other μ_j. All of these results are remarkably explicit.

The sequence $\{\mu_j\}$ will be the eigenvalues of a Sturm-Liouville operator with zero boundary conditions only if the asymptotic distribution $\mu_j \sim j^2\Pi^2$ holds. This follows from the behavior of the solution of the initial value problem $y(0,\lambda) = 0$, $y'(0,\lambda) = 1$, $-y'' + qy = \lambda y$. As $|\lambda| \to \infty$, $y(x,\lambda)$ approaches the solution for $q = 0$, namely $\sin(\sqrt{\lambda x})/\sqrt{\lambda}$, and the eigenvalues μ_j are the roots of the entire function $y(1,\lambda)$, so $\mu_j \sim j^2\Pi^2$. With a bit more work, one may show that if $q \in L^2$, $\mu_j - j^2\Pi^2 - \int_0^1 q(x)\,dx$ is a square-summable sequence.

Viewing μ_n as a function of $q \in L^2$, one may also compute its gradient. This is the function $\partial\mu_n/\partial q(t)$ such that

$$\frac{d}{d\varepsilon}\mu_n(q+\varepsilon p)|_{\varepsilon=0} = \int_0^1 \frac{\partial\mu_n}{\partial q(t)} \cdot p(t)\,dt$$

for all $p \in L^2$. A formal calculation, which may be rigorously justified, shows that $\partial\mu_n/\partial q(t) = g_n^2(t)$, where $g_n(t)$ is the normalized eigenfunction $(\int_0^1 g_n^2(t)\,dt = 1,\ g_n'(0) > 0)$ for the eigenvalue μ_n. Formally, solving

$$-g_n'' + (q+\varepsilon p)g_n = \mu_n(q+\varepsilon p)\,g_n$$

to first order in ε (the linearized equation) gives an equation for the derivative of g_n with respect to q in the direction p. With z denoting that derivative, the equation is:

$$-z'' + qz + pg_n = \mu_n z + \frac{d}{d\varepsilon}\mu_n(q+\varepsilon p)|_{\varepsilon=0}\,g_n.$$

Multiplying by g_n, integrating, and using the fact that the differential operator is selfadjoint, the first two terms of the left-hand side cancel with the first term on the right, leaving

$$\int_0^1 pg_n^2(t)\,dt = \frac{d}{d\varepsilon}\mu_n(q+\varepsilon p)|_{\varepsilon=0} \quad \text{or} \quad \frac{\partial \mu_n}{\partial q(t)} = g_n^2(t).$$

The appearance of squared eigenfunctions in perturbation calculations is central to the methods of the book and well known to experts in related contexts.

Once squared eigenfunctions arise, it is desirable to evaluate certain L^2 inner products. A curious algebraic fact about products of eigenfunctions underlies this evaluation.

LEMMA. *If y_1 and y_2 both satisfy $-y'' + qy = \lambda y$ for the same λ, then the product $y_1 y_2$ satisfies the skew-adjoint equation*

$$-z''' + 4qz' + 2q'z = 4\lambda z'.$$

This follows by a direct calculation. This shows in turn that g_n^2 is perpendicular in L^2 to $(d/dx)(g_m^2)$ for $m \neq n$.

Geometrically speaking, the elements $(d/dx)(g_m^2)$ are perpendicular to the gradients of each of the μ_j's so they are (formally) tangent vectors to the set of all elements in L^2 giving fixed values of $\{\mu_j\}$. This suggests that the subset of L^2 given by a particular level set of μ_j's is in fact infinite-dimensional.

Remarkably, one may find global coordinates for this set of "isospectral potentials". These are given by the numbers

$$\kappa_n(q) = \log|y'(1,\mu_n)| = \log|g_n'(1)/g_n'(0)| \quad \text{for } n \geq 1,$$

where $g_n(x)$ is again the normalized eigenfunction. Asymptotically,

$$\kappa_n(q) = \frac{1}{2\pi n}\int_0^1 q(t)\cdot\sin(2\pi n t)\,dt + O(1/n^2)$$

so that $n\kappa_n$ is square-summable if $q \in L^2$. The gradient may also be computed. Thus the combined mapping

$$q \mapsto (\{\kappa_n(q)\}, [q], \{\mu_n(q) - n^2\Pi^2 - [q]\})$$

is an isomorphism of L^2 onto an open subset of $l_1^2 \times \mathbf{R} \times l^2$, where l^2 denotes the Hilbert space of square-summable sequences and l_1^2 denotes the Hilbert space of sequences $\{\kappa_n\}$ such that $\{n\kappa_n\}$ is square-summable.

Once these coordinates are known, the vector fields which generate the straight-line flows (with μ fixed) $\kappa \to \kappa + t\xi$ for $\xi \in l_1^2$ may be pulled back to determine vector fields on L^2 which move the κ coordinates without changing the Dirichlet spectrum $\{\mu_j\}$. It is also easy to show that $\kappa = 0$ corresponds to a potential q symmetric about $x = 1/2$, so this shows that *every spectrum may be realized by a unique potential q which is symmetric about $x = 1/2$.*

Another remarkable fact is that the solution curves to these differential equations are expressible in an essentially closed form. The solution to $\dot{q} = 2(d/dx)(g_n^2(x;q))$ with initial value $q_0(x)$ is given by

$$q_0(x) - 2\frac{d^2}{dx^2}\log\left[1 + (e^t - 1)\int_x^1 g_n^2(s;q_0)\,ds\right].$$

The form $-2(d^2/dx^2)\log(\theta_n(x,t,q_0))$ *arises from* the algebra connected with factoring the Sturm-Liouville operator. This goes back at least to Darboux (some authors also refer to Bäcklund, as well as d'Alembert) and is both elementary and beautiful.

Namely, if g is a nontrivial solution of $-y''+qy=\mu y$ and f is a nontrivial solution of $-y''+qy=\lambda y$ for $\lambda \neq \mu$, then for the same value of λ, the new operator

$$-y'' + \left(q - 2\frac{d^2}{dx^2}\log g\right) y = \lambda y$$

has nontrivial solution $(gf'-g'f)/g$, while for $\lambda = \mu$, the new operator has a general solution $(a + b\int_0^x g^2(s)\,ds)/g$ for arbitrary a, b. In interpreting the operator, consider only those points where $g \neq 0$. The factorization of $-d^2/dx^2 + q - \mu$ into

$$-\left[\frac{1}{g}\left(\frac{d}{dx}\right)(g)\right]\cdot\left[g\frac{d}{dx}\frac{1}{g}\right]$$

follows from the equation for g. One then computes the permuted product

$$\begin{aligned}
-\left[g\frac{d}{dx}\frac{1}{g}\right]\left[\frac{1}{g}\frac{d}{dx}g\right] &= -\frac{d^2}{dx^2} - \frac{g''}{g} + 2\left(\frac{g'}{g}\right)^2 \\
&= -\frac{d^2}{dx^2} + \frac{g''}{g} - 2\frac{d^2}{dx^2}\log g \\
&= -\frac{d^2}{dx^2} + \left(q - 2\frac{d^2}{dx^2}\log g - \mu\right).
\end{aligned}$$

Thus a solution f of $-(d^2/dx^2)f + qf - \lambda f$ leads to a solution $\tilde{f}$ of

$$-\frac{d^2}{dx^2}\tilde{f} + \left(q - 2\frac{d^2}{dx^2}\log g - \lambda\right)\tilde{f} = 0,$$

where

$$\tilde{f} = \left(g\frac{d}{dx}\frac{1}{g}\right) f = g\left(\frac{f'}{g} - \frac{fg'}{g^2}\right) = \frac{1}{g}(gf' - g'f),$$

while $1/g$ is annihilated by $(1/g)(d/dx)g$.

To complete the solution of the inverse problem, Pöschel and Trubowitz show how to construct vector fields which move one eigenvalue μ_j while fixing all the others. These vector fields have solutions locally in time, up to the "collision" of two eigenvalues. Again the generators are essential derivatives of products of eigenfunctions. This is used to show that any admissible spectrum may be realized.

Inverse spectral theory is a clearly written, complete account of the solution of the inverse problem for a Sturm-Liouville operator with Dirichlet boundary conditions. Exercises are mixed in nicely, so that the exposition still flows but the reader can be self-tested along the way. The appendices help to fill in needed background, especially in functional analysis. All of these reflect the origin of the book, a one-semester course on this topic.

As the authors admit, the subject is presented in isolation. In the preface, Trubowitz suggests that the contents of a second volume would include the periodic problem (where $q(x)$ is periodic in x). This would then bring in the Korteweg–de Vries flows, algebraic geometry, and a wealth of beautiful material. This reviewer would welcome such a book written with the same care and polish as the current volume and would gladly teach a course based on it.

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