BOOK REVIEWS

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Three-dimensional link theory and invariants of plane curve singularities, by David Eisenbud and Walter Neumann. Annals of Mathematics Studies, vol. 110, Princeton University Press, Princeton, N.J., 1985, vii + 171 pp., $13.95 (paper), $39.50 (hardback).

A knot is an embedded circle in the three-sphere. The simplest example of a knot is the ordinary overhand knot or trefoil (Figure 1); another is the figure eight knot (Figure 2). The trefoil and figure eight are clearly different (not isotopic), and both are obviously knotted (not isotopic to the unknotted circle or unknot (Figure 3)).

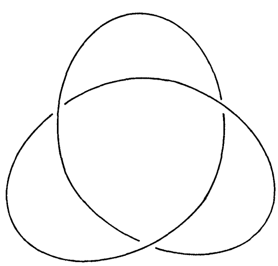

FIGURE 1

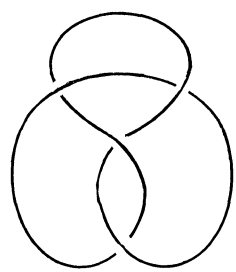

FIGURE 2

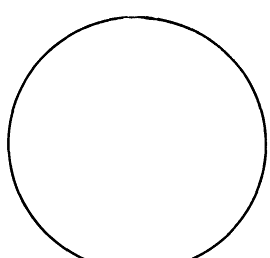

FIGURE 3

It is difficult, however, to prove this. Knots were in fact one of the testing grounds of early twentieth-century topology. A number of invariants were invented to distinguish knots. One such invariant is the Alexander polynomial; for the trefoil, this polynomial is $t^2 - t + 1$, for the figure eight it is $t^2 - 3t + 1$ and for the unknot it is just 1. Hence these two knots are different, and both are knotted. Another invariant is the fundamental group of the complement of a knot. There are classical algorithms for computing this group, which tends to be rather large, and its many invariants, one of which is the Alexander polynomial mentioned above. More complicated is the study of a link, an embedded collection of circles in the three-sphere. The complication comes from the fact that the individual circles can not only be knotted, but also linked with each other. Many of the invariants of knots have more complicated analogues for links, together with strenuous algorithms for their computation.

A complex plane algebraic curve is the locus of zeros in two-dimensional complex space of a polynomial of two variables. In general, this locus will be a smooth one-dimensional complex manifold, but at certain isolated points it may have a singularity. The topological structure of these singularities has been known for some time, and is not too complicated. Consider a small four-ball in the complex plane with center the singular point of the curve. This four-ball is of course a cone on its boundary, a three-sphere. The curve intersects this three-sphere in a smooth (real) one-dimensional object, which is thus a knot or link in this three-sphere. Furthermore, inside the ball the curve is topologically the cone on this intersection. Thus singularities of plane curves give links; these are called algebraic links. The topological invariants of these discrete objects are essential to the algebraic study of curve singularities. Indeed there are classical algorithms connecting the polynomial, the resolution of the singularity, and the structure of the link. Some of these algorithms are beautiful, some less so; almost all are messy, and easily forgotten.

It is difficult to imagine that much more could be said about this subject. However, during the last twenty years there have been some new developments. Three-manifold topology has made tremendous advances, mainly due to the work of Thurston. There is now a conjectural decomposition of a three-manifold into pieces, each of which has a geometric structure; these geometric structures are of eight fairly well-understood types. Another surprising discovery had been made earlier by Milnor, who found that the complement of an algebraic link is a fiber bundle over the circle, the fibers being two-manifolds with boundary the link. Hence a number of new invariants appeared, and these needed to be related to the decomposition described above as well as the invariants of curve singularities described earlier. Straightening out the old material, explaining the new and fitting both together thus provided ample subject matter for a book. Still, this process could have been routine; but fortunately Eisenbud and Neumann found beautiful connections among these topics and contributed new ideas as well.

The fundamental construction on links presented in the book is called splicing. This concept is quite easy to define: First, the class of links should be enlarged to include not just those in ordinary spheres, but in homology spheres as well. (A homology sphere is a topological space with the homology of a sphere.) Given two links, together with distinguished components of these links, their splice is formed by removing tubular neighborhoods of the distinguished components and gluing the spaces that remain together, matching meridian to longitude and longitude to meridian. The result is a new link in a new homology sphere. This simple operation, generalized from work of Siebenmann, has probably been overlooked since the new ambient space may be a nontrivial homology sphere even if the original ambient spaces are ordinary spheres. It turns out that many common operations on links such as connected sum and cabling are in fact special cases of splicing. The authors note that the conjectural decomposition of Thurston mentioned earlier is then a theorem in this case (work of Jaco-Shalen and Johannson): An irreducible link is the result of splicing together a collection of links, each of whose complements is Seifert fibered or simple. Furthermore, this splicing is essentially unique. (According to Thurston, in most cases the complement of a simple link has a hyperbolic structure. For example, the complement of the trefoil knot is Seifert fibered, and the complement of the figure eight knot has a hyperbolic structure.) A pleasant property of splicing is that most link invariants, for example the Alexander polynomial, are additive in an appropriate sense.

Eisenbud and Neumann are mainly concerned with links they call "graph links"; these are links whose splice components have Seifert fibered structures. The class of these links includes the class of algebraic links mentioned earlier. For this type of link they give a handy graph-like notation, a 'splice diagram', with special kinds of vertices and edges. Vertices with an arrowhead, for example, correspond to components of the link, and those with an open circle correspond to a Seifert manifold embedded in the link exterior. The diagram makes computation of many link invariants easier: Linking numbers of components of the link can be computed by tracing a path in the diagram between the corresponding vertices, the fibering of the link complement becomes a condition on the diagram, and so forth.

Another useful concept introduced in this book is that of 'multilink'. A multilink is just an ordinary link with an integer attached to each link component. The integer is to be interpreted as a multiplicity; this concept is useful since the splice components of a link are multilinks, and furthermore, a singularity of a nonreduced curve gives a multilink.

The book contains much pleasant geometry. For Seifert links, that is, links whose complements have Seifert fibrations, it is shown (with one exception) that the whole homology sphere is Seifert fibered and that the link is a collection of fibers. Two descriptions are given of these Seifert-fibered homology spheres: The first is the standard topological description as a circle bundle over a punctured two-sphere with singular fibers, and the second is as the link of a certain type of singularity, a Brieskorn complete intersection. This leads to a clearer picture of how the Seifert fibering by circles fits together with the Milnor fibering by surfaces.

Knot theory is a rich subject because of its many readily available examples, and this book is not an exception to the rule. There are examples of links which are not graph links (and an elegant proof via linking numbers that they are not); examples, originally due to Grima, of distinct fibered links with the same Alexander polynomial, and examples of nontrivial links with trivial fibering of the complement. Most of these are easily computed using the splice diagram.

One general pattern in the study of topological properties of algebraic varieties is that some property is first discovered using the algebraic structure, and later reproved in a geometric or topological setting. Eisenbud and Neumann do this as well: They show geometrically that the complement of algebraic link fibers, the eigenvalues of the monodromy on homology are roots of unity, and that its largest Jordan block associated to the eigenvalue one is of size one. (The first two facts already had geometric proofs; the last one had been proved analytically by Steenbrink and later Navarro, using mixed Hodge theory. They also obtained a related result in higher dimensions.)

My only major complaint with the book is that there is no index. An index should be easy to make in our present computer age (and was easy even in the past, when a student could be hired to do it). The lack of an index reduces the value of a book as a reference. I suspect that an index would help to organize such matters as definitions as well. For example, the important term 'solvable link' appears to be defined in the statement of Theorem 9.2, and the central term 'graph link' appears not to be formally defined at all; it is first mentioned in the body of the text in §8, without definition, but fortunately had already been defined—in passing—in the introduction. The many favorable qualities of this book more than make up for these defects, though. It is an excellent book for students to read, especially because of its collection of scattered and occasionally unknown results, for example the algorithm for passing from the Puiseux expansion to the polynomial. And lastly, there is a lengthy and superb introduction.

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Methods of representation theory with applications to finite groups and orders, vol. II, by Charles W. Curtis and Irving Reiner. John Wiley and Sons, New York, Chicester, Brisbane, Toronto, Singapore, 1987, xv+951 pp., $95.00. ISBN 0-471-88871-0

"... while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with the properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations."