at home on personal computers. It really would not be that difficult, starting from the database the editors of this work have painstakingly assembled. In other words, what we have here are six lovely oranges, but what I want is just one Apple.

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Differential geometry of complex vector bundles, by Shoshichi Kobayashi. Publications of the Mathematical Society of Japan, no. 15 Iwanami Shoten Publishers and Princeton University Press, Princeton, N. J., 1987, xi+304 pp., $57.50. ISBN 0-691-08467-x

The book under review is a research monograph laying the foundation for the theory of Einstein-Hermitian structures on holomorphic vector bundles. The concept of an Einstein-Hermitian structure has been introduced by the author in 1978 and has proved to be very fundamental and popular since. Being fundamental usually is not sufficient for being popular; what made this concept so popular? I see at least two principal reasons:

The first is that it provides a link between differential geometry and algebraic geometry, leading to a good problem, the so-called Kobayashi-Hitchin conjecture. This problem has in the meantime been completely solved by Donaldson [**2**, **3**] and Uhlenbeck and Yau [**10**].

The second reason is that the solution of this conjecture in the 2-dimensional case made several spectacular applications possible.

In complex dimension 2 the conjecture ties Yang-Mills theory and algebraic geometry. Together with Donaldson's fundamental work on instanton moduli spaces it led to unexpected results on the differential topology of algebraic surfaces [**4**, **5**, **9**].

What is an Einstein-Hermitian structure? To explain this consider a compact complex submanifold $X \subset \mathbf{P}^N_{\mathbf{C}}$ of some projective space endowed with the induced metric. A holomorphic vector bundle $\mathcal{E}$ on X is a locally trivial fibre space over X with fibres $\mathbf{C}^r$ and holomorphic transition functions. Suppose we want to equip $\mathcal{E}$ with a Hermitian structure h, i.e., a C^∞ family $(h(x))_{x \in X}$ of Hermitian metrics on the fibres. It would then be natural to look for a best or in some sense distinguished structure h. Now it is a fundamental fact that every choice of a Hermitian structure on a holomorphic bundle gives rise to an associated concept of parallelism, in other words, to a compatible connection D_h in $\mathcal{E}$. The mean curvature K_h of this connection is a Hermitian form on $\mathcal{E}$, also depending on the metric on X, which measures how the bundle is twisted. If this form is proportional to the metric h, $K_h = c \cdot h$, one says that h is an Einstein-Hermitian structure or that $(\mathcal{E}, h)$ is an Einstein-Hermitian bundle.

This concept obviously generalizes the notion of a Kähler-Einstein metric. Kobayashi arrived at the definition of an Einstein-Hermitian structure when he was working on vanishing theorems for holomorphic tensor fields, but he realized immediately that there is a close relation between his Einstein condition and the concept of a stable vector bundle in algebraic geometry, a notion that ultimately goes back to Mumford. In fact, he and independently Lübke [**6**] proved that irreducible Einstein-Hermitian bundles are stable in the sense of Mumford and Takemoto [**8**]. The question whether the converse of this is true has been posed by Hitchin and Kobayashi. It became known as the Kobayashi-Hitchin conjecture.

The first situation in which this conjecture was known to be true was the case of Riemann surfaces. Here it follows essentially from an old result of Narasimhan and Seshadri [**7**] as reformulated by Atiyah and Bott [**1**]. The next and most important case was due to Donaldson [**2**]; he proved the conjecture for projective algebraic surfaces. Now in complex dimension 2 the Einstein condition coincides with the anti-self-duality condition in Yang-Mills theory (if $\mathscr{E}$ has trivial determinant); but anti-self-duality is a notion that can be formulated for connection in bundles over any oriented Riemannian 4-manifold X. These anti-self-dual connections, the so-called instantons, had played a role in elementary particle physics for some time. Their equivalence classes form finite-dimensional spaces, the instanton moduli spaces. It was the fundamental observation of Donaldson that these instanton spaces carry information about the differential topology of the 4-manifold X [**4**]. In order to extract this information one needs a way to compute these spaces explicitly. For projective algebraic surfaces this is provided by the solution of the Kobayashi-Hitchin conjecture, which translates the problem into a problem in algebraic geometry, where it can be solved under good circumstances. One specific result that has been obtained in this way is that the very innocent looking surface $\widetilde{\mathbf{P}}^2(x_1, \ldots, x_9)$, the projective plane blown up in 9 points, admits infinitely many different C^∞-structures [**5**, **9**].

The general case of the Kobayashi-Hitchin conjecture, i.e., the case of stable bundles on compact Kähler manifolds, has recently been solved by Uhlenbeck and Yau [**10**].

Kobayashi's book arose out of several series of lectures and seminars that the author gave between 1981 and 1985 at the Universities of Tokyo, Tsukuba and Berkeley. It has been written for graduate students and researchers in various fields of mathematics, including algebraic and differential geometry, complex analysis, low-dimensional topology and even mathematical physics.

The first chapter of the book provides a carefully written introduction to the theory of connections in complex vector bundles; the second discusses Chern classes and their representation in terms of curvatures. The material in these two chapters is fairly standard; it should be of interest to a wider audience.

The third chapter treats vanishing theorems for holomorphic vector bundles under certain negativity conditions, such as negativity of the mean curvature. Some of these theorems play a central role in the following sections on Einstein-Hermitian structures.

In Chapters IV and V the author lays the foundation for the theory of Einstein-Hermitian bundles. He characterizes Einstein-Hermitian structures as the absolute minima of a Yang-Mills type functional and proves a fundamental inequality for their Chern classes.

After a discussion of Einstein-Hermitian bundles over Riemann surfaces he recalls some general facts about coherent sheaves and stable vector bundles and then shows that irreducible Einstein-Hermitian bundles are always stable. Chapter VI discusses a weak form of the Kobayashi-Hitchin conjecture. It can be viewed as an introduction to Donaldson's paper [**2**], in which he solves this conjecture for projective algebraic surfaces. Donaldson's proof is not included in the book.

The final Chapter VII contains recent results, mostly due to the author and his coworkers on moduli spaces of Einstein-Hermitian structures and simple holomorphic bundles. The level of exposition is not uniform throughout the book. While the first 5 chapters assume only standard knowledge of complex manifolds, the technical requirements increase in the later part. The last two chapters demand at least some background in analysis. My only complaint about the book is that it was finished at an unfortunate time, namely right before the complete solution of the Kobayashi-Hitchin conjecture was available. It is to be hoped that this solution will be included in a possible future edition. In any case, this excellent monograph will certainly become a classic, just as the previous books of the author.

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