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Decompositions of manifolds, by Robert J. Daverman. Academic Press, Orlando, 1986, xi+317 pp., $55.00. ISBN 0-12-204220-4

Upper semicontinuous decompositions were introduced by R. L. Moore in 1924 [**9**]. The first major result of the theory was Moore's theorem [**10**] that if G is an upper semicontinuous decomposition of the plane $\mathbf{R}^2$ into compact connected nonseparating sets, then the associated decomposition space is topologically equivalent to the plane.

During the next twenty-five years or so, the theory of upper semicontinuous decompositions was developed extensively, especially by Moore and members of his school. This theory was applied to the study of continua and locally connected continua, the plane and 2-manifolds, mappings of special types (open, monotone, and light in particular), and dimension theory.

Generalizing the Moore theorem on decompositions of $\mathbf{R}^2$ has been a goal of the theory of upper semicontinuous decompositions. A number of the major results of the theory are of the following type: If G is an upper semicontinuous decomposition of a space X, and G and X have certain properties, then the associated decomposition space is homeomorphic to X.

In the nineteen fifties there was considerable work on trying to establish a theorem for $\mathbf{R}^3$ analogous to Moore's theorem on decompositions of $\mathbf{R}^2$. G. T. Whyburn [**12**] had suggested in 1936 the property (named later) of being pointlike as a candidate to replace nonseparation in the plane case. A compact connected pointlike set in $\mathbf{R}^3$ behaves homeomorphically like a point. But in the fifties, Bing discovered his famous dogbone decomposition of $\mathbf{R}^3$ [**3**], a decomposition of $\mathbf{R}^3$ into compact connected pointlike sets for which the decomposition space is topologically distinct from $\mathbf{R}^3$.

Beginning with Bing's work in the fifties, there has been a major emphasis in decomposition theory on the study of manifolds. This theory has been developed extensively and has contributed in a major way to the development of geometric topology, especially the study of the structure and properties of manifolds, and embedding theory. It has also yielded a number of major results in topology. Among these are the following:

(1) The existence of nonstandard periodic homeomorphisms of $\mathbf{R}^n$ and S^n for $n \geq 3$ [**1**, **2**].

(2) The existence of nonmanifold factors of $\mathbf{R}^n$ for $n > 3$ [**3**].

(3) The generalized Schönflies Theorem [**4**].

(4) The proof that if H^n is a homology n-sphere, its double suspension $\Sigma^2 H^n$ is an $(n+2)$-sphere [**5**, **6**].

In addition, Freedman's solution of the 4-dimension Poincaré conjecture [**7**] made essential use of this theory.

The foundations for the current theory were laid by Bing with his work, especially that of the fifties. Several of the fundamental ideas and techniques are due to him, and his influence still shapes the growth of the theory. Although Bing's work dealt primarily with $\mathbf{R}^3$, much of it could be readily extended to $\mathbf{R}^n$ and, with some technical changes, to n-manifolds.

In his early work, Bing concentrated on pointlike decompositions of $\mathbf{R}^3$. By the sixties, with Brown's proof of the generalized Schönflies Theorem [**4**] and other results, it became clear that for manifolds, the correct idea was that of a cellular decomposition. In $\mathbf{R}^n$, cellular and pointlike are equivalent, and in manifolds, cellular sets behave topologically like points.

As the theory developed in the mid-sixties, its scope widened to include certain aspects of the theory of retracts. Under this influence, as well as that of the then-emerging shape theory, attention was focussed on a more general type of set. These are the cell-like sets, sets which behave homotopically like points.

This is the setting for Daverman's book where the main objects of study are upper semicontinuous decompositions of finite dimensional manifolds into cell-like sets, or for short cell-like decompositions of manifolds. Daverman concentrates on dimensions greater than 4, since in this dimension range, unified and powerful topological techniques are available. In contrast, each of the lower dimensions has its own special techniques.

We shall now give an outline of the contents of Daverman's book. Chapter 1 is devoted to basic definitions and essential facts about proper maps.

In certain cases, we may regard decompositions and mappings as interchangeable. Given a surjective map $f: X \to Y$, there is an induced decomposition of X, $\{\{f^{-1}(y): y \in Y\}$. Given a decomposition G of X, there is a surjective map $f: X \to X/G$, the projection map.

Both of these aspects of the theory are useful in practice, and there are advantages in being able to switch one's point of view. Sometimes to study a known space Y, it is useful to give a description of Y as the decomposition space X/G of some space X by an upper semicontinuous decomposition G of X. Identifying Y as X/G often involves constructing a map from X to Y that mimics the projection map from X to X/G. On the other hand, one of the most powerful uses of decompositions is in the construction of entirely new spaces.

Chapter 2 is long and introduces the key technical idea of the book, shrinkability of a decomposition, an idea due originally to Bing. It is a consequence of one of the basic theorems of the subject that if G is an upper semicontinuous decomposition of $\mathbf{R}^3$ into compact sets and there is a proper map $f: \mathbf{R}^3 \to \mathbf{R}^3$ that induces G, then $\mathbf{R}^3/G$ is homeomorphic to $\mathbf{R}^3$. Bing's insight was that in some case, such an f can be obtained as the limit of a sequence of homeomorphisms which shrink nondegenerate elements of G more and more, in a controlled way.

Bing first used this shrinking technique to show that the union of two solid Alexander horned balls, glued together by the identity on their boundaries, is a 3-sphere [**1**]. This construction easily yields a nonstandard involution on S^3.

In Chapter 2, the notion of cellularity is introduced, and the generalized Schönflies Theorem [**4**] proved. Several results of this chapter are extensions to $\mathbf{R}^n$ of results, many due to Bing, originally obtained for $\mathbf{R}^3$. There are several results concerning decompositions of $\mathbf{R}^n$ with only countably many nondegenerate elements. This chapter includes a number of historically important examples of cellular decompositions of $\mathbf{R}^3$.

Bing discovered [**3**] that if X is his dogbone decomposition space, then $X \times \mathbf{R}$ is topologically $\mathbf{R}^4$. This result has been one of the major motivating results for decomposition theory in the ensuing years. In Chapter 2, there are a number of results concerning products of decomposition spaces with the real line $\mathbf{R}$.

By combining results on squeezing 2-cells to arcs with a clever idea due to Giffen [**8**], Daverman constructs a homology 3-sphere H^3 whose double suspension is a 5-sphere. Historically this result was discovered after Edwards had proved [**6**] that the double suspension of any homology 3-sphere is a 5-sphere, but this example is important for its relative simplicity. Such examples are important since they yield noncombinatorial triangulations of S^5.

Chapter 3 concerns cell-like sets. Retracts provide a natural environment for the study of such sets and various generalizations.

Decompositions of compact metric spaces into compact connected sets can raise dimension. However, whether this can happen for cell-like decompositions of compact metric spaces is still unknown, and is one of the major open questions of the theory.

Chapter 4 is devoted to the proof of Edward's cell-like approximation theorem. According to this result, a cell-like decomposition of an n-manifold, $n \geq 5$, is shrinkable if and only if G satisfies a certain technical condition dealing with continuous images of 2-cells in M^n/G. This result, the culmination of efforts by Edwards, Cannon, and others, is quite powerful, and forms much of the technical basis for the remainder of the book. Since Edward's proof is (as yet) otherwise available, the inclusion of this proof makes Daverman's book especially valuable.

Chapter 5 deals with methods of constructing decompositions, including taking products, spinning, and slicing. A major open question in the theory of cell-like decompositions is whether $(\mathbf{R}^n/G) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^{n+1}$ when G is a cell-like decomposition of $\mathbf{R}^n$. This question was considered in Chapter 2 for certain special types of decompositions, and in Chapter 5, it is studied in greater generality.

In Chapter 6, the emphasis is on constructing examples of nonshrinkable decompositions. Some of the examples described are related to examples in $\mathbf{R}^3$ described earlier. Considerable attention is paid to a quite general notion of defining sequence.

The final chapter is devoted to applications of decomposition theory to the study of manifolds, especially generalized manifolds. A particularly important question, almost completely solved by Quinn [**11**] is whether each generalized n-manifold is the cell-like image of some (topological) manifold.

The book is organized as a textbook with numerous exercises, of varying degrees of difficulty, and there are even some open questions stated. The first half of the book provides a thorough introduction to the subject. Proofs are given in detail, and a good background in elementary geometric topology should be sufficient for this part of the book. The second half of the book brings the reader to the forefront of research in the field. Here the pace is brisker, and a familiarity with piecewise linear techniques is desirable.

Daverman's book is the first devoted exclusively to the theory of decompositions. It is much needed, and provides an excellent treatment of a subject of growing importance.

REFERENCES

1. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

2. ——, *Inequivalent families of periodic homeomorphisms of E^3*, Ann. of Math. (2) **80** (1964), 78–93.

3. ——, *The Cartesian product of a certain nonmanifold and a line is E^4*, Ann. of Math. (2) **70** (1959), 399–412.

4. M. Brown, *A proof of the generalized Schönflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

5. J. W. Cannon, $\Sigma^2 H^3 = S^5/G$, Rocky Mountain J. Math. **8** (1978), 527-532.

6. R. D. Edwards, *Suspensions of homology spheres* (unpublished manuscript).

7. M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

8. C. H. Giffen, *Disciplining dunce hats in 4-manifolds* (unpublished manuscript).

9. R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proc. Nat. Acad. Sci. **10** (1924), 356–360.

10. ——, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), 416–428.

11. F. Quinn, *Resolutions of homology manifolds, and the topological characterization of manifolds*, Invent. Math. **72** (1983), 267–284; Corrigendum, Invent. Math. **85** (1986), 653.

12. G. T. Whyburn, *On the structure of continua*, Bull. Amer. Math. Soc. **42** (1936), 49–73.

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Hamiltonian methods in the theory of solitons, by L. D. Faddeev and L. A. Takhtajan. Translated by A. G. Reyman. Springer-Verlag, Berlin, Heidelberg and New York, 1987, ix+592 pp., $110.00. ISBN 3-540-15579-1

The modern theory of integrable or solvable systems was initiated by the discoveries of Gardner, Greene, Kruskal, Miura and Zabusky in their investigations of the Korteweg-de Vries equation during the sixties. There then followed a period of intensive activity, which lasted until the late seventies, during which the characteristic features of these systems were explored and a vast class of such equations discovered. It is fair to say that many of the major advances in this field are associated with groups of researchers at a particular institute, such as the Leningrad group to which the authors of this book belong.

Most of the solvable equations possess a family of special solutions, which can be obtained in closed form. In the simplest cases, such as the Korteweg-de Vries equation, they can be given a physical interpretation as a collection of interacting particles. Each particle has only nearest neighbour interaction