ON REAL ALGEBRAIC MODELS OF SMOOTH MANIFOLDS

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An affine nonsingular real algebraic variety $X$ diffeomorphic to a smooth manifold $M$ is said to be an algebraic model of $M$. The remarkable theorem of Nash-Tognoli asserts that each compact smooth manifold $M$ has an algebraic model [16 or 6, Theorem 14.1.10]. In fact, there exists an infinite family $\{X_i\}_{i \in \mathbb{N}}$ of irreducible algebraic models of $M$ such that $X_i$ and $X_j$ are birationally nonisomorphic for $i \neq j$ [10] (cf. also [7] for a proof in a special case). In view of these results, it seems natural and interesting to investigate algebro-geometric properties of various algebraic models of a given smooth manifold. This paper addresses a few questions of this type. For notions and results of real algebraic geometry we refer the reader to the book [6].

Given a compact affine nonsingular real algebraic variety $X$, denote by $H_{k}^{\text{alg}}(X, \mathbb{Z}/2)$ the subgroup of $H_k(X, \mathbb{Z}/2)$ of the homology classes represented by (Zariski closed) $k$-dimensional algebraic subvarieties of $X$ [6, Chapter 11 or 11]. Let $H_{k}^{\text{alg}}(X, \mathbb{Z}/2)$ be the image of $H_{k}(X, \mathbb{Z}/2)$, $d = \dim X$, under the Poincaré duality isomorphism $H_{d-k}(X, \mathbb{Z}/2) \rightarrow H_k(X, \mathbb{Z}/2)$. Although the groups $H_{k}^{\text{alg}}(X, \mathbb{Z}/2)$ are one of the most important invariants of $X$ (a sample of applications can be found in [1, 2, 3, 6, 8, 9]), our knowledge of their behavior is still rather limited. Here we consider the following.

**Problem.** Let $M$ be a compact smooth manifold and let $G$ be a subgroup of $H_{k}(M, \mathbb{Z}/2)$. When are there an algebraic model $X$ of $M$ and a diffeomorphism $\varphi: X \rightarrow M$ such that the induced isomorphism

$$\varphi^*: H_{k}(M, \mathbb{Z}/2) \rightarrow H_{k}(X, \mathbb{Z}/2)$$

maps $G$ onto $H_{k}^{\text{alg}}(X, \mathbb{Z}/2)$?

This problem has attracted the attention of several mathematicians (cf. [3, 4, 5, 6, 12, 14, 15]), however, the results are far from complete. We have a solution for $k = 1$, $M$ connected, and $\dim M \geq 3$.

**Theorem 1.** Let $M$ be a compact connected smooth manifold with $\dim M \geq 3$ and let $G$ be a subgroup of $H_{1}(M, \mathbb{Z}/2)$. Then the following conditions are equivalent:

(i) There exists an algebraic model $X$ of $M$ and a diffeomorphism $\varphi: X \rightarrow M$ such that $\varphi^*(G) = H_{1}^{\text{alg}}(X, \mathbb{Z}/2)$.

(ii) The first Stiefel-Whitney class $w_1(M)$ of $M$ is in $G$.

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In particular, if $M$ is orientable, then (i) is always satisfied.

For smooth surfaces and $k = 1$ we have only a partial, but quite satisfactory, solution. First let us define the following invariants of a compact nonsingular real algebraic surface $X$:

$$
\beta(X) = \dim H^1_{\text{alg}}(X, \mathbb{Z}/2),
$$

$$
\delta(X) = \dim \{ v \in H^1_{\text{alg}}(X, \mathbb{Z}/2) \mid v \cup v = 0 \}.
$$

If $X$ is connected, orientable (resp. nonorientable of odd topological genus), then $\beta(X) = \delta(X)$ (resp. $\beta(X) = \delta(X) + 1$); indeed, $w_1(X)$ is in $H^1_{\text{alg}}(X, \mathbb{Z}/2)$ [6, Theorem 12.4.8] and $w_1(X) \cup w_1(X) \neq 0$. For $X$ connected, nonorientable of even topological genus, one has either $\beta(X) = \delta(X)$ or $\beta(X) = \delta(X) + 1$ and, in accordance with Theorem 2 below, all topologically possible cases can be realized algebraically.

**Theorem 2.** Let $M$ be a compact connected smooth surface of genus $g$.

(i) If $M$ is orientable (resp. nonorientable of odd genus) and $\beta$ is an integer satisfying $0 \leq \beta \leq 2g$ (resp. $1 \leq \beta \leq g$), then there exists an algebraic model $X_\beta$ of $M$ with $\beta(X_\beta) = \beta$.

(ii) If $M$ is nonorientable of even genus and $\beta$ and $\delta$ are integers satisfying either $\beta = \delta$, $1 \leq \beta \leq g - 1$, or $\beta = \delta + 1$, $2 \leq \beta \leq g$, then there exists an algebraic model $X_{\beta,\delta}$ of $M$ such that $\beta(X_{\beta,\delta}) = \beta$ and $\delta(X_{\beta,\delta}) = \delta$.

Our interest in the invariants $\beta(X)$ and $\delta(X)$ is explained by the fact that they determine the projective module group $K_0(\mathcal{R}(X))$ of the ring $\mathcal{R}(X)$ of regular functions from $X$ to $\mathbb{R}$.

**Theorem 3.** (i) Let $X$ be a compact connected affine nonsingular real algebraic surface. Then

$$
K_0(\mathcal{R}(X)) \cong \mathbb{Z} \oplus (\mathbb{Z}/4)^{\beta(X) - \delta(X)} \oplus (\mathbb{Z}/2)^{\beta(X) + 1 - 2(\beta(X) - \delta(X))}.
$$

(ii) As $X$ runs through all algebraic models of a compact, connected surface $M$ of genus $g$, then the groups $K_0(\mathcal{R}(X))$ take, up to isomorphism, precisely $q(M)$ values, where

$$
q(M) = \begin{cases} 
2g + 1 & \text{if $M$ is orientable,} \\
g & \text{if $M$ is nonorientable and $g$ is odd,} \\
2g - 2 & \text{if $M$ is nonorientable and $g$ is even.}
\end{cases}
$$

Condition (i) is proved in [9], while (ii) follows immediately from (i) and Theorem 2.

Here is another application. Given a compact affine nonsingular real algebraic variety $X$, let $C^\infty(X, S^1)$ denote the topological group of $C^\infty$ mappings from $X$ to the unit circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ (the group structure on $C^\infty(X, S^1)$ is induced from that on $S^1$ and the topology is the $C^\infty$ one). Let $\mathcal{R}(X, S^1)$ be the closure in $C^\infty(X, S^1)$ of the subgroup $\mathcal{R}(X, S^1)$ of regular mappings from $X$ to $S^1$. Below we are concerned with the quotient group

$$
\Gamma(X) = C^\infty(X, S^1)/\mathcal{R}(X, S^1).
$$
Theorem 4. Let $M$ be a compact connected smooth manifold with $\dim M \geq 2$. Let

$$r_M: H^1(M, \mathbb{Z}) \otimes \mathbb{Z}/2 \to H^1(M, \mathbb{Z}/2)$$

be the canonical monomorphism and let

$$\alpha(M) = \begin{cases} \text{rank } H^1(M, \mathbb{Z}) - 1 & \text{if } M \text{ is nonorientable,} \\ \text{rank } H^1(M, \mathbb{Z}) & \text{otherwise.} \end{cases}$$

(i) For each algebraic model $X$ of $M$, one has $\Gamma(X) \cong (\mathbb{Z}/2)^s$ for some $s$ with $0 < s \leq \alpha(M)$.

(ii) For each integer $s$ satisfying $0 < s \leq \alpha(M)$, there exists an algebraic model $X_s$ of $M$ with $\Gamma(X_s) \cong (\mathbb{Z}/2)^s$.

Sketch of Proof. Let $X$ be a compact affine nonsingular real algebraic variety. By [8, Theorem 1.4], a mapping $f$ in $C^\infty(X, S^1)$ is in $Z(X, S^1)$ if and only if $f^*(H^1(S^1, \mathbb{Z}/2))$ is contained in $H^1(X, \mathbb{Z}/2)$. It follows that $\Gamma(X)$ is isomorphic to the quotient group $A/B$, where $A = \text{Image } r_X$ and $B = A \cap H^1(X, \mathbb{Z}/2)$. This implies (i) and, applying Theorems 1 and 2, also (ii).

Proofs of Theorems 1 and 2 are quite involved. Here we can only sketch the proof of Theorem 1.

The implication (i) $\Rightarrow$ (ii) is well known (cf. [6, Theorem 12.4.8]). Suppose then that (ii) holds. One easily finds a $C^\infty$ embedding $i: M \to P$ of $M$ into the product $P = \mathbb{R}P^{k_1} \times \cdots \times \mathbb{R}P^{k_r}$ of real projective spaces with $i^*(H^1(P, \mathbb{Z}/2)) = G$. It requires some care to construct a compact smooth surface $S$ in $M$ such that $j^*(v) \neq 0$ for all $v$ in $H = H^1(M, \mathbb{Z}/2) \setminus G$, where $j: S \to M$ is the inclusion mapping, and $S$ bounds a compact smooth submanifold $W$ of $M$ with the property that the normal vector bundles of $i(W)$ in $i(M)$ and $P$ are trivial. Using the triviality of the normal vector bundle of $i(W)$ in $P$, one shows the existence of a $C^\infty$ diffeomorphism $h: P \to P$, arbitrarily close in the $C^\infty$ topology to the identity mapping, such that $Y = h(i(S))$ is a nonsingular algebraic subvariety of $P$ (cf. [12, §2]). Moreover, and this is the hard part of the construction, $h$ can be chosen in such a way that $H^1_{\text{alg}}(Y, \mathbb{Z}/2) = 0$. To achieve this, one uses, in particular, appropriate real algebraic versions of the theorem of Gherardelli [13, Theorem 6.5] and the theorem of Noether-Lefschetz-Moishezon [13, Theorem 7.5] concerning the Picard group of complex projective varieties.

Since $H^*_\text{alg}(P, \mathbb{Z}/2) = H^*(P, \mathbb{Z}/2)$ and $Y$ has trivial normal vector bundle in $N = h(i(M))$, it follows from [1, Proposition 2.8] that there exist a positive integer $q$ and a $C^\infty$ embedding $e: N \times \{0\} \to P \times \mathbb{R}^q$, arbitrarily close in the $C^\infty$ topology to the inclusion mapping $N \times \{0\} \to P \times \mathbb{R}^q$, such that $X = e(N \times \{0\})$ is a nonsingular algebraic subvariety of $P \times \mathbb{R}^q$ containing $Y \times \{0\}$. Clearly, $\varphi: X \to M$, defined by the condition $e(h(i(\varphi(x)))) = x$ for $x$ in $X$, is a $C^\infty$ diffeomorphism and $H^1_{\text{alg}}(X, \mathbb{Z}/2)$ contains $\varphi^*(G)$. Since $H^1_{\text{alg}}(Y \times \{0\}, \mathbb{Z}/2) = 0$ and $f^*(\varphi^*(v)) \neq 0$ for all...
\(v\) in \(H\), where \(f: Y \times \{0\} \to X\) is the inclusion mapping, it follows that 
\(\varphi^*(G) = H^1_{\text{alg}}(X, \mathbb{Z}/2)\), i.e., (i) is satisfied.

**REFERENCES**