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Multiple forcing, by Thomas Jech. Cambridge Tracts in Mathematics, vol. 88, Cambridge University Press, Cambridge, 1986, vii + 136 pp., $34.50. ISBN 0-521-26659-9

In set theory truth is approached from four directions, not just the usual two. A given proposition may be true or false, but it may also be consistent with or independent of "the usual axioms for set theory", that is to say the Zermelo-Fraenkel system of axioms together with the Axiom of Choice, the whole denoted by ZFC. Moreover, these independent propositions take up part of the life of every set theorist. They are not the sort of propositions that only a logician could love; frequently they are powerful, fundamental assertions that occur naturally, and they require study. To follow this Fourfold Way of Truth one must master, in addition to proof and refutation, the method of forcing.

Forcing, of course, was invented 25 years ago by Paul Cohen as the key element in his proof that the Continuum Hypothesis is independent of ZFC. It can best be regarded as a way of adjoining to the universe of set theory new sets with special properties. For example, to make the Continuum Hypothesis false one might adjoin $\aleph_2$ new real numbers. Now from the point of view of the universe V of set theory any new sets have got to be fictitious, since V is nothing else than the collection of all (well-founded) sets, so there is a flavor of sand-castle-building to the whole enterprise of forcing. One way of dealing with this is to treat the extension of the universe as a collection of artificial constructs, "fuzzy" sets if you like, in which the truth values of propositions are taken from a complete Boolean algebra. The "ordinary" universe then corresponds to the case of the trivial two-element Boolean algebra. It is more common, however, to take a supra-universal view of forcing, to think of forcing as an operation performed on one universe of set theory, the *ground model*, to produce another, the *Cohen extension.*

A notion of forcing designed to construct a new set A typically consists of "conditions" giving partial information about A. The set P of conditions is partially ordered by extension, where p extends q if p gives more information about A than q (and p agrees with the information given by q). The set A is then equiconstructible with the set of all $p \in P$ giving correct information about A. The latter set is called a generic set for P; it is a maximal consistent set of information satisfying certain other requirements as well.

Although forcing is a child of mathematical logic its applications are remarkably logic-free. For example, the assertion that a combinatorial proposition holds in the Cohen extension generally translates into the assertion that the partial ordering of forcing conditions has some combinatorial property in the ground model, and this tends to be an accessible problem for a couple of reasons. First, one can choose the set of conditions. Properties that are difficult to check for one set may be quite easy for another. Second, one can choose the ground model. Gödel showed that if ZFC is consistent then so is ZFC together with the Axiom of Constructibility, which implies the Generalized Continuum Hypothesis (GCH). Thus without loss of generality one can assume that GCH holds in the ground model, so cardinal arithmetic is vastly simplified and combinatorial properties are easier to check. Of course, as a library of additional propositions consistent with ZFC is accumulated, the number of potential ground models is multiplied. The ground model for one notion of forcing may be the Cohen extension obtained from another.

The most important corollary of the discovery of forcing has been the recasting of the goals of set theory. Once upon a time, that is to say before Cohen, the purpose of set theory was considered to be the investigation of the universe of all sets in the same sense that number theory is the investigation of the collection of all natural numbers. Unfortunately, many of the deepest and most interesting questions of set theory have turned out to be undecidable in ZFC. Now this in itself is not a problem, for we cannot expect ZFC to capture the whole truth about the universe of sets any more than the axioms of Peano Arithmetic capture the whole truth about the natural numbers. But propositions going beyond these axiomatic systems have got to be either true or false. Gödel's famous sentence may not be provable but it is at least *true.* Among the natural elementary propositions of number theory known to be independent of Peano Arithmetic there is no doubt which ones are true—usually, it must be admitted, because they are provable in a stonger system like ZFC—but the situation is quite the opposite for the Continuum Hypothesis. There seem to be no intuitive grounds on which one can make a case for the truth or falsity of CH. Worse, the intuition is all on the side of forcing. When we force over a model of CH to get a Cohen extension in which CH fails, and then force over that model to get a further extension in which CH is true again (just add a new set, a bijection between the reals and the countable ordinals) there is a clear and strong intuition that leads us to believe we see what is really going on. This is exactly the kind of intuition we would like to have about the truth or falsity of CH, but it is about independence instead.

This yields a very strong temptation to convert set theory from an absolute enterprise like number theory to a relative enterprise like group theory. Perhaps "true" shouldn't mean "true in the universe of all sets" but rather "true in all models of ZFC" or "true in all universes" just as a theorem of group theory is a statement true of all groups. Perhaps we shouldn't be asking *whether* CH is true or false, but *how* it can be true or false.

That leads to the problem of the classification of universes, with forcing as the tool for passing from one universe to another, and that is the view of set theory we get in Thomas Jech's book *Multiple forcing*. Jech is well known to students of set theory as the author of the standard reference in the field **[1]**, which already contains a thorough treatment of forcing. In this slender new book the focus is on forcing done more than once, for example forcing to adjoin many real numbers or forcing iterated transfinitely.

There is a nice section on different ways to adjoin real numbers. One can find definitions and in most cases proofs of elementary properties of Cohen reals, Sacks reals, Prikry-Silver reals, Laver reals, Grigorieff reals and random reals (now, unfortunately, often called Solovay reals). To add enough reals to violate CH it is generally necessary to form the product of at least $\aleph_2$ copies of the partial ordering for adding one real and to endow the product with finite support, countable support, or some kind of mixed support in order to be sure that all cardinal numbers in the ground model remain cardinals in the Cohen extension. The intuition is that with product forcing the reals are added all at once, or "side by side," as opposed to being added one at a time, iteratively. The two constructions are quite different in general. In addition to the standard facts, the treatment of product forcing includes such gems as that adding two random reals (or two Laver or Mathias reals) side by side automatically adds a Cohen real, whereas this does not happen when the reals are added iteratively. This is quite possibly the only published reference dealing with all these forcing notions.

Not surprisingly, iterated forcing is usually employed when it is necessary to handle many potential counterexamples one at a time. Consider, for example, the Borel Conjecture. A set A of real numbers has *strong measure zero* provided that for any sequence $\{\varepsilon_n\}$ of positive reals, there is a sequence $\{I_n\}$ of intervals so that I_n has length ε_n for each n, and $A \subseteq \bigcup_{n=1}^{\infty} I_n$. The Borel Conjecture states that every set of strong measure zero must be countable. It is false under CH, and a fairly easy forcing argument shows that it is consistently false when CH fails. The problem is to show that it is consistently true. Given an uncountable set A of strong measure zero, the natural way to "kill" it is by adjoining a new sequence $\{\varepsilon_n\}$ of reals for which no corresponding sequence $\{I_n\}$ of intervals exists.

Unfortunately it is quite possible that adjoining the sequence $\{\varepsilon_n\}$ will inadvertently produce a new uncountable set of strong measure zero, which must then be killed separately, and so on. We are led to force iteratively, killing every counterexample we encounter. This raises two fundamental problems. First, will we ever finish? There has to be some point by which we have considered every possible counterexample. Second, is it possible that a counterexample killed by an earlier forcing may be brought back to life by a later one? The later forcing might accidentally produce a sequence $\{I_n\}$ that satisfies the sequence $\{\varepsilon_n\}$ adjoined earlier. These two problems are typical of iterated forcing.

Jech treats Laver's consistency proof for the Borel Conjecture, and in the same part of the book he discusses the Solovay-Woodin consistency proof for Kaplansky's conjecture and Shelah's argument for the Whitehead problem. Kaplansky's conjecture asserts that every homomorphism on $C[0, 1]$ into any commutative Banach algebra is continuous. Whitehead's question is whether every W-group is free, where a W-group is an infinite abelian group such that for every epimorphism $\pi: B \to A$ with kernel $\mathbf{Z}$ there is a homomorphism $\varphi: A \to B$ such that $\pi \circ \varphi$ is the identity.

There is not room in 130 pages to give complete accounts even of these three problems, much less all the other topics found in this book. Jech's approach is to try to isolate those lemmas critical to the idea of the proof, and then to steer the reader to the literature for the rest of the details. I think this is rather effective; it will be particularly helpful for the student who knows some forcing but is intimidated by a literature that does not always make the important ideas clear.

Shelah deduces a negative answer to Whitehead's question from a powerful proposition known as Martin's Axiom (MA). (Is MA true? Well, Jech shows *how* to make it true.) MA is an internal forcing axiom, i.e., the assertion that for a class C of partial orderings, every member of C possesses partially generic sets that are internal in the sense that they belong to the universe (the ground model, not some Cohen extension). For MA, C is the class of orderings with the countable chain condition. The last third of the book is mostly devoted to two more powerful cousins of MA, the Proper Forcing Axiom and Martin's Maximum (MM). Not only are the internal forcing axioms very powerful, they are also relatively easy to use. That makes them popular with many people, like set-theoretic topologists, who need the power of independence techniques without the aggravation of learning all the details.

The consistency proof for MM, due to Foreman, Magidor and Shelah, is iterated forcing's finest hour. It brings together state-of-the-art iteration methods with large-cardinal arguments. The consequences of MM, like the fact that $2^{\aleph_0} = \aleph_2$ and that the nonstationary ideal on ω_1 is $\aleph_2$-saturated, are simply stunning, and the implications of the methods used to prove it consistent are still being worked out. Like the other topics in this fine book, Jech's treatment of it is incomplete but concentrated on the important points. The reader who works all the way through this book, filling in the gaps, will have learned a great deal of set theory and will be left on the very edge of research.

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Dimensions of ring theory, by Constantin Nastasescu and Freddy van Oystaeyen. D. Reidel Publishing Company, Dordrecht, Boston, Lancaster and Tokyo, 1987, xi + 360 pp., $74.00. ISBN 90-277-2461-x

While mathematics is certainly not a "science of measurement", mathematicians do seek to "take the measure" of everything they study, often in the form of numerical, cardinal, or ordinal invariants—for example, to measure the deviation of a certain system from some ideal situation, to measure how likely or unlikely a certain object is to enjoy a certain property, or simply to measure the progress in some inductive procedure. Many such invariants have evolved, either directly or through analogy, from the Euclidean dimensions with which we measure our "real" world, and thus many invariants are called "dimensions" of some sort, usually decorated with one or more adjectives. Ring theory has its share of such dimensions, attached to both rings and modules, and since these dimensions have evolved in an algebraic rather than a geometric environment, their connection with Euclidean dimension may not be readily apparent. To illustrate, we discuss three examples—Goldie dimension, Krull dimension, and Gelfand-Kirillov dimension.

Goldie dimension. Vector space dimension cannot be applied directly to arbitrary modules because most modules do not have bases, and even among those that do (namely the free modules), one can find modules in which different bases may have different cardinalities. The dimension of a vector space V can, however, be expressed as the number of terms in a decomposition of V into a direct sum of irreducible subspaces, or as the maximum number of terms occurring in decompositions of V into direct sums of nonzero subspaces. Since the complexity of a module need not be reflected by direct sum decompositions, one looks at decompositions of submodules along with decompositions of a given module. Thus the *Goldie dimension* (also called the *uniform dimension*, the *uniform rank*, or the *Goldie rank*) of a module M is defined to be the supremum of the number of nonzero terms in any direct sum decomposition of any submodule of M.

This dimension arose in Goldie's 1958 development of noncommutative rings of fractions [3], since one necessary condition for a ring R to have a simple artinian ring of fractions Q is that the Goldie dimension of R (considered as a module over itself) be finite. More specifically, such a Q must be isomorphic (by the Artin-Wedderburn Theorem) to the ring