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Radically elementary probability theory, by Edward Nelson. Annals of Mathematical Studies, vol. 117, Princeton University Press, Princeton, N. J., 1987, ix + 97 pp., $40.00 ($15.00 paperback). ISBN 0-691-08473-4

The titles of few mathematics books are as bold and eye catching as this one. Does a radical proposal with perhaps illicit pleasures wait inside? Yes. As a replacement for the conventional measure theoretic foundations of probability, Nelson proposes elementary finite probability spaces and a "tiny bit of nonstandard analysis." Nonstandard analysis provides an enriched class of *finite* objects including both the *infinitely large* and the *infinitely small.* With these new *finite* objects it is possible to study a rich variety of probabilistic phenomena within the framework of finite probability spaces. These are the proposed new foundations. The book, *Radically elementary probability theory*—here after REPT—is an example of what can be done.

The book is beautifully written and essentially self-contained. A remarkably simple introduction to the necessary nonstandard analysis is presented in 13 pages. The remainder of the text develops a basic graduate course in stochastic processes. The subject matter focuses mainly on martingales and the "*Wiener walk*". Starting on p. 1 with the definition of a random variable the path goes through nonstandard analysis, martingales, the law of large numbers, the central limit theorem, the Lévy-Doob martingale characterization of the Wiener process, and the invariance principle. It is all done in 79 pp.

The treatment, although elementary, is not familiar. Theorems which look familiar are different enough to cause considerable discomfort. The uninitiated reader may be very unclear about the content, generality and flexibility of the methods and theorems. The questions of content and generality are addressed quite successfully in an appendix to which we will return later. The questions of flexibility and generality of the methods will, I think, only be settled in time but there is clearly more life in finite probability than many suspected, and there is much here that will reward careful study. The remainder of our review attempts to describe Nelson's contribution in nonstandard analysis and expands some on the changes suggested for probability.

Nonstandard analysis was created by Abraham Robinson in 1960 to lay a rigorous foundation for infinitesimals in analysis. The first step in this program was to build a new (nonstandard) enlargement ($^*\mathbb{R}$) of the real numbers $\mathbb{R}$, which included the sought after infinitesimals. The structure of $^*\mathbb{R}$ was required to mirror that of $\mathbb{R}$. Two important machines, the "transfer principle" and the "standard part maps" were built to allow free movement between the standard and nonstandard worlds.

Robinson's success was spectacular. One can already see in the first edition of his book [9] which appeared in 1966, that a very powerful tool had been created. It produced both new theorems and significant simplifications in the proofs of many results. More important however was the fact that nonstandard analysis provided new concepts, viewpoints and intuitions that were not available in conventional mathematics. The world of analysis was richer and more interesting after the inclusion of infinitesimals than before.

For the general mathematical community this development involved a large amount of difficult mathematical logic, but fortunately this was soon reduced by the discovery that Łos' theorem on ultrapower extensions readily implied the transfer theorem. This version of nonstandard analysis has become the orthodox version, here after called ONSA, and is now a mature highly developed technical subject.

ONSA is contributing significantly to current developments in probability, see e.g. Anderson [**2**] on the construction of Brownian motion. Keisler [**5**] on stochastic analysis, Loeb [**6**] on the construction of an important class of probability spaces (now called Loeb spaces), and Perkins' work [**8**] on Brownian local times. There are good expositions of ONSA, see especially [**1** and **4**] that provide both a manageable entry into the subject and a wealth of applications.

In 1977 Nelson published an alternative nonstandard analysis which he called internal set theory or IST [7]. This article explicitly states that it is aimed at implementing Robinson's statement "...from a formalist point of view we may look at our theory syntactically and may consider what we have done is to introduce new deductive procedures rather than new mathematical entities." [**9**, p. 282] Nelson takes this syntactic approach. Rather than using the technical constructions of ONSA he adopts an axiomatic approach and achieves surprising results.

The axiomization begins with the introduction of a new undefined term, the predicate *standard*. A formula is called *internal* if it does not use the term standard, and is called *external* otherwise. Three axiom schemes are introduced that describe the rules for dealing with the new term *standard*, and we are warned never to use external predicates to define sets. These axioms are added to classical mathematics (more precisely to ZFC = Zermelo-Fraenkel set theory with the axiom of choice) and nothing else is changed. Classical mathematics was simply enriched by adding new terms and rules of deduction. The resulting theory of IST contains classical mathematics. If ZFC is consistent then so is IST, and every internal statement which can be proved in IST can also be proved in ZFC. This situation is described by saying that IST is a conservative extension of ZFC.

However, the introduction of IST brings a radical change in the mathematical perspective. All specific objects of classical mathematics such as $\sqrt{2}$ and $SL(2, \mathbb{Z})$ are standard in IST. The real numbers $\mathbb{R}$ and the natural numbers $\mathbb{N}$ are unchanged. A set B is *finite* iff there is no bijection of B onto a proper subset of B. However, there are *nonstandard* objects. In fact, each infinite set contains *nonstandard* elements so there are *nonstandard* n in $\mathbb{N}$ This happens even though 0 is *standard* and if n is *standard* then $n+1$ is *standard*. The point here is that the induction principle holds for subsets of $\mathbb{N}$, but there is no set $S = \{n \in \mathbb{N}\colon n \text{ is } \textit{standard}\}$. Remember the warning never to use *external* predicates to define sets. Only *internal* relations may be used to define sets!

The following are some basic facts about IST. Each *nonstandard* $n \in \mathbb{N}$ is larger than any *standard* $m \in \mathbb{N}$ but n is still finite and may be used in any formula like $\sum_{j=1}^{n} j$ or $n!$. For $n \in \mathbb{N}$, the set $\{m \leq n: m \in \mathbb{N}\}$ is finite but is standard iff n is standard. A real number x is an *infinitesimal* if $|x| \leq 1/n$ for some *nonstandard* $n \in \mathbb{N}$. The basic elements of nonstandard analysis, the infinitely large and small quantities which in ONSA appeared as elements of ${}^*\mathbb{R}$, are in IST simply elements of $\mathbb{R}$. Classical mathematics is enriched here through new concepts and procedures, but not through new constructs.

When compared with ONSA, the axiomatic method of IST is simpler, more elementary, and is easier to learn, but it lacks some of the flexibility of ONSA. This is particularly true as concerns the external sets which exist in ONSA, but are absent from IST. Basic constructions in ONSA such as the nonstandard hull of a Banach space and Loeb measures are not part of IST unless the constructions are paraphrased or the simple theory is modified and made more complex. Concerning possible modifications see [7], also [3].

In the book REPT the nonstandard analysis is an axiomatic IST development, but is simplified and entirely confined to the natural numbers and $\mathbb{R}$. The basic new axioms are (1) 0 is *standard*, (2) there is an n in $\mathbb{N}$ that is *not standard*, (3) for all n in N if n is standard then $n+1$ is *standard* and (4) if $A(0)$ and if for all n whenever $A(n)$ then $A(n+1)$, then for *standard* n we have $A(n)$. A fifth less essential axiom was occasionally used. It is truly remarkable how far one can go with these axioms. In particular, they support Nelson's entire elementary development of stochastic processes.

An appendix relates the elementary theory to the conventional theory. It presents several large (nonelementary) theorems which show that the elementary theorems imply their conventional analogues. Thus the elementary approach is available to prove conventional theorems. This however is not Nelson's aim. The elementary theory is simpler and has the "same scientific content as the conventional theory." His conclusion is that the elementary theory stands on its own without being translated back into the more elaborate conventional language. A monograph illustrating how elementary nonstandard analysis can be used to do conventional mathematics would not be radical. It is radical to suggest that the translation be skipped and that the mathematics be left in its elementary finite form.

This book shows that Nelson's suggestions have much to offer probability. A plausible explanation for this is that in probability the intuition and conceptual foundations are often essentially finite and combinatorial and are usually much simpler than the elaborate technical theorems which result. I do not expect that these elementary foundations will displace either the conventional approach to probability or the growing use of ONSA within probability. I do expect that REPT and the research that it generates will greatly add to the development of probability and to our understanding of its foundations.

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Stability in model theory, by Daniel Lascar. Translated by J. E. Wallington. Pitman Monographs and Survey in Pure and Applied Mathematics, vol. 36, Longman Scientific and Technical, Essex, 1987, 193 pp., $77.95. ISBN 0-470-20386-2

Officially, model theory was created in 1950 by Alfred Tarski **[Ta]**, who described it as a subject that lies on the "borderline between algebra and metamathematics." Many see it, nowadays, as generalized algebra, a point of view that is strongly emphasized in this book.

There is one obvious connection between algebra and the concept of model. The familiar classes of algebraic structures are usually described as the class of *models* of a given list of axioms. Such are, for instance, the class of groups, the class of fields, the class of algebraically closed fields. In these examples and in many more, the axioms can be stated as *first order sentences* in a suitable language. Let us explain what we mean by this. A logical language L comes equipped with a supply of operation and relation symbols of given arities (thus, we have a language for groups, another one for fields, a third one for ordered fields, etc.); one of them is always the equality symbol '='. The first order sentences of L are statements that use these symbols as well as variables and are constructed by means of logical connectives ("not", "and", "or", etc.) and quantifiers ("for all x", "there exists x", where x is a variable). The variables are required to be of the