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Hilbert modular surfaces, by Gerard van der Geer. Ergebnisse der Mathematik, Band 16, Springer-Verlag, Berlin, Heidelberg, New York, 1988, ix + 291 pp., $69.50. ISBN 3-540-17601-2

The theory of Hilbert modular surfaces is a generalisation of the classical theory of automorphic forms, and in many ways it is one of the easiest generalisations. It was started by D. Hilbert at the end of the last century, with the motivation to enhance the theory of analytic functions of several complex variables. He inspired O. Blumenthal to take up the subject, and sometimes his name is added to that of Hilbert. However, neither man got very far, and only after the general theory of complex manifolds had advanced sufficiently could progress be made in this special case. In modern times the subject has been revived by M. Rapoport and F. Hirzebruch. The theory of Hilbert modular surfaces mixes the theory of automorphic forms, arithmetic algebraic geometry (especially Shimura-varieties) and the theory of classical complex algebraic surfaces. It thus seems appropriate to give short overviews of recent developments in these subjects, and after that we try to explain how they specialize to the case of Hilbert modular surfaces. Needless to say, I tend to oversimplify the situation; for details one should consult the literature.

The theory of automorphic forms started with the classical elliptic modular functions (for a modern account see [**La**]), and has developed into a theory about reductive groups. Let us try to explain how this happened: Classically one considers the upper halfplane $\mathbf{H}$ of complex numbers with positive imaginary part, on which the group $\mathrm{SL}(2,\mathbf{R})/\{\pm 1\}$ acts by the usual $(az+b)/(cz+d)$-rule. One further chooses a subgroup Γ of finite index in $\mathrm{SL}(2,\mathbf{Z})$, and considers holomorphic functions $f(z)$ which transform under Γ according to a certain factor of automorphy, and which are holomorphic at infinity (this amounts to a certain growth-condition).

There is a distinction between two types of automorphic forms, Eisenstein-series and cusp-forms: The former are certain rather well-understood series, which define automorphic forms and are characterised by their asymptotics at infinity. The latter are automorphic forms which vanish at infinity, and their properties are much more mysterious. The whole space of automorphic forms is the direct sum of the cusp-forms and the Eisenstein-series.

On these spaces there is an operation by Hecke-operators: For any element $\gamma \in \mathrm{GL}(2, \mathbf{Q})$ the intersection of Γ and $\gamma\Gamma\gamma^{-1}$ has finite index in Γ, and the γ-transform of an automorphic form is invariant under this subgroup. Summing over these transforms under a set of representatives for Γ modulo the subgroup gives a new automorphic form, and this way we obtain a huge algebra (the Hecke-algebra) operating on the space of automorphic forms. Now one decomposes everything into eigenspaces under this algebra, which usually have dimension one.

One can generalise this by considering nonholomorphic functions (which however are still eigenfunctions for the Laplacian). The whole theory then becomes very group-theoretic: The upper half-plane $\mathbf{H}$ is a symmetric space, namely the quotient $\mathbf{H} = G/K$, with $G = \mathrm{SL}(2, \mathbf{R})$ and $K = \mathrm{SO}(2, \mathbf{R}) =$ maximal compact subgroup of G. Then functions on $\mathbf{H}/\Gamma$ correspond to K-invariant functions on the G-homogeneous space G/Γ, and automorphic forms to K-eigenfunctions. Automorphic forms in the previous sense now correspond to irreducible constituents (under the G-operation) in the space of functions on G/Γ: Such constituents always contain certain canonical K-eigenvectors, which are the automorphic forms in the previous sense. For example, holomorphic automorphic forms are related to discrete series representations.

Finally one passes to an adelic formulation: Instead of $\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$ we consider $\mathrm{SL}(2, \mathbf{A})/\mathrm{SL}(2, \mathbf{Q})$, where $\mathbf{A}$ is the ring of adeles, and we replace $\mathrm{SO}(2, \widehat{\mathbf{R}})$ by the product $\mathrm{SO}(2, \mathbf{R}) \times \mathrm{SL}(2, \widehat{\mathbf{Z}})$, which is a maximal compact subgroup in $\mathrm{SL}(2, \mathbf{A})$. We try to decompose the space of functions on $\mathrm{SL}(2, \mathbf{A})/\mathrm{SL}(2, \mathbf{Q})$ into irreducible (under $\mathrm{SL}(2, \mathbf{A})$) components. These are restricted tensorproducts of irreducible representations of the local groups $\mathrm{SL}(2, \mathbf{R})$ respectively $\mathrm{SL}(2, \mathbf{Q}_p)$. The representation of $\mathrm{SL}(2, \mathbf{R})$ is our previous one, while the representations of the factors $\mathrm{SL}(2, \mathbf{Q}_p)$ describe the combinatorics of the Hecke-operators. It turns out that this very much simplifies these combinatorics as well as the rest of the theory, mainly since the structure of the group $\mathrm{SL}(2, \mathbf{Q})$ is much simpler than that of $\mathrm{SL}(2, \mathbf{Z})$, because of the Bruhat-decomposition. Finally we have to admit that for a good theory it is better to replace SL(2) by GL(2). If the reader has been willing to follow us so far he can now proceed to read the lecture notes of Jacquet-Langlands or Weil [**JL** and **W**], which give the general theory of automorphic forms on GL(2) over any numberfield (not just $\mathbf{Q}$). In general one tries to replace GL(2) by an arbitrary reductive algebraic group G over $\mathbf{Q}$, and one wants to decompose $G(\mathbf{A})/G(\mathbf{Q})$. Again there is a decomposition into cusp-forms and Eisenstein-series. The latter are associated to forms on groups of lower dimension, and should be handled by induction (which still poses quite a variety of problems, see [**L2**]).

This leaves cusp-forms, and it is quite hard to understand them. There are general but mostly unproven conjectures by Langlands relating them to Galois-representations. The only general method of attack presently known is the Selberg-trace formula: This expresses the trace of certain operators on the space of cusp-forms (on $G(\mathbf{A})/G(\mathbf{Q})$) as a sum of terms parametrised by conjugacy classes in $G(\mathbf{Q})$. In principle these traces determine the representation. However the formulas tend to be so complicated that the best one can do is to compare them to similar complicated objects. This way one sometimes can find relations between cusp-forms on different groups, by comparing the corresponding trace-formulas. For example one obtains a relation between forms on GL(2) and on units of quaternion-algebras (see **[JL]**), or between GL(2)'s over different fields **[L1]**. However it seems fair to say that for general groups the technical problems posed by the trace-formula still have not been completely mastered.

In the case of Hilbert modular surfaces we deal with the case of GL(2) over a real quadratic numberfield. Here the technicalities of the trace-formula are understood, and there are also alternate means to construct and classify automorphic forms. So from the point of view of automorphic forms many basic problems have been solved, and we can hope to use this knowledge to interact with other aspects of the situation.

The second relevant point of view is that of Shimura-varieties. Classically the quotient $Y(\Gamma) = \mathbf{H}/\Gamma$ is a Riemann-surface which is algebraic, that is by adding finitely many points (the cusps) one obtains a compact Riemann surface $X(\Gamma)$. This surface is an algebraic curve, so that it can be described as the common set of zeroes, in complex projective space, of finitely many homogeneous polynomials. Moreover it turns out that $X(\Gamma)$ is already defined over a numberfield, that is the coefficients of the homogeneous polynomials above can be chosen in this numberfield. Finally one can even give models over the integers of this field. All this follows easily from the fact that $Y(\Gamma)$ parametrises elliptic curves (with some "level-structure"), which gives an intrinsic algebraic definition. As the relevant moduli-problem is defined over the integers of some numberfield this must also hold for the moduli-space itself.

In higher dimensions the quotients of hermitian symmetric spaces under arithmetic groups are again quasiprojective algebraic varieties, which can be compactified to normal projective algebraic varieties (the Baily-Borel compactification) or to complex algebraic manifolds where infinity is a divisor with normal crossings (the toroidal compactification). Again one can show that these are defined over certain specific numberfields (although this is much harder now), but there seems to be no general method to get models over the integers. However many important examples are moduli-spaces of certain types of abelian varieties (usually with some condition on their endomorphisms), which easily gives models over numberfields and sometimes over their integers. These have played an important role for testing certain general conjectures about varieties over numberfields. For example, by counting the number of points in finite fields (for this we need an integral model) one can define the zetafunction of an algebraic variety. This is an analytic function of a complex variable s, defined by an infinite

Euler-product which converges in some right halfplane $\mathrm{Re}(s) \gg 0$, and one conjectures that it extends meromorphically to the whole complex plane, and satisfies a functional equation. For some Shimura-varieties this can be checked (following ideas of Langlands) by relating the zetafunction to automorphic forms. The technical tools involved are the trace-formula and the combinatorial description of points over finite fields.

In the case of Hilbert modular surfaces all these techniques apply: They form the moduli-space for abelian surfaces with multiplication by the integers of the real quadratic field, which defines a moduli-space over the integers. Also the counting techniques and the trace-formula work, so that the zetafunction is in fact given by automorphic forms. Finally one can check the Tate-conjecture, which claims that the order of vanishing of the zetafunction at a certain s-value is given by the rank of the Néron-Severi group of the surface, which in turn should be determined by the Galois representation of the second l-adic cohomology-group. An important tool here is the fact that one naturally has a whole bunch of curves on these surfaces.

Finally there is the theory of complex algebraic surfaces. These have been classified around the turn of the century. (For a modern account see **[BPV]**.) To classify a given surface one first has to construct a minimal model, by blowing down exceptional curves. Then one should compute the Kodaira-dimension, which can take the values $-\infty, 0, 1$ or 2. If this is different from 2 some further invariants suffice to determine the structure of the surface (like being rational, or a K3-surface, etc.), by looking into a suitable table. (As usual for classifications the subject has a botanical flavour.) However for Kodaira-dimension 2 (the surfaces of general type) the classification is rather incomplete, that is we do not know very much about the structure of such surfaces. In this sense the classification still needs some refining.

For Hilbert modular surfaces one naturally tries to locate them in this zoo of algebraic surfaces. First it is already an interesting problem to determine a minimal model, which can be solved except if the surface is of general type (which however most of them are). Then one has to determine numerical invariants, which are usually given by some number-theory, for example by special values of L-series. Finally one follows the procedure outlined above.

Another special feature of Hilbert modular surfaces is the relation to singularities. In general a Hilbert modular surface has singularities at infinity (which are resolved by the toroidal compactification) and at the fixed-points where the group does not operate freely. The latter are quotient-singularities and again allow an explicit resolution (this is much more difficult in higher dimensions). This way one obtains very interesting examples of surface-singularities and their resolutions. This third aspect of the theory (that is Hilbert modular surfaces as special examples of complex algebraic surfaces) has been very much the subject of F. Hirzebruch's work in the field during the seventies.

Now as we have explored the territory we can try to locate the book in it. In short it can be said that it follows very much the Bonn school, which is not surprising considering the biography of the author. Most of it deals with Hilbert modular surfaces over the complex numbers: The author gives their definition, resolves the singularities at the cusps and at the elliptic fixed points, computes global invariants, introduces the curves F_N and T_N on them which are given naturally via the modular interpretation, and relates the singular cohomology to modular forms. After that the surfaces are classified, examples are given, and the embedding of Hilbert modular surfaces into the Siegel threefold (classifying principally polarised abelian surfaces) is studied. So far the book follows very much in style and content the work of F. Hirzebruch. At the end it gets more arithmetic, as the proof of the Tate-conjecture (due to Harder, Langlands and Rapoport) is described. All in all it is well written but by no means selfcontained, as sometimes arguments from the general theory which are not specific to the subject are left to the reader. So it can be easily understood by somebody with a good general knowledge about complex surfaces, but probably not by a beginner in the field. Also on occasion there are slight lapses in the proofs, which provides the reader with some exercises (the only ones in the book). As tradition demands that the reviewer has to find at least one mistake we give here as an example the liberal use of Kodaira vanishing on p. 72.

However, all in all I enjoyed my reading, and recommend the book to anybody interested in the subject.

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